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COMPUTATION

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A TREATISE

ON

COMPUTATION

AN ACCOUNT OF THE CHIEF METHODS FOR CONTRACTING AND ABBREVIATING ARITHMETICAL CALCULATIONS

605-24

BY

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"*He who can easily, rapidly, and accurately, add, subtract, multiply and divide, is a computer.*"—DE MORGAN

"*Superfluitas impedit multum . . . et reddit opus abominabile.*"—
ROGER BACON

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PREFACE.

In the following little treatise I have aimed at giving an account of the chief general methods for arriving at rapidity of numerical calculation. I have in nearly all cases tried to show the nature of general methods by actual examples worked, rather than by sets of verbal rules, believing strongly in the Newtonian maxim, "*Exempla magis prosunt quam precepta*".

If due attention were paid in schools to arithmetic as an art of computation, a considerable part of what is here put down might have been omitted; but the majority of teachers seem to be contented to go on inculcating cumbrous and antiquated methods of work which must be discarded by any one who has to perform rapidly and accurately such calculations as are necessary in many branches of applied science; and I have had to point out that fundamental changes must be made in the methods of performing even the most elementary operations.

Section I. is therefore devoted to explaining generally what methods are to be chosen for effecting various arithmetical operations without raising the question of *abbreviation* or *approximation.*

Section II. shows how to abbreviate the processes employed, to obtain approximate results, and to estimate the amount of error to which these are liable. Attention is drawn to the *essentially approximate nature* of physical calculations, and to the consideration, sometimes neglected, that to whatever degree of accuracy the *arithmetical operations* are carried out, there cannot be a higher degree of accuracy in the *results obtained* than in *the numerical data on which those calculations are founded.*

Section III. deals with logarithms. Specimens of various trigonometrical tables are given, and their uses explained. In particular, I have insisted on the use of cologarithms, and on the omission of the tabulated 10 from the logarithms of trigonometrical ratios.

Section IV. illustrates the principles previously explained by applying these to such calculations as actually occur in the office or the laboratory. I have to thank Professor Ayrton for the free use he has allowed me to make of the examples in his *Practical Electricity*.

In a work involving such a large number of figures there is only too good reason to suspect some undiscovered errors in the printing. I shall be grateful to any one who, on detecting any such error, will inform me of it.

In addition to these, I cannot help suspecting others, both of omission and commission, for which I am alone responsible. For all these I ask the indulgence of my readers, and venture to hope that they will help me in the task I have undertaken by pointing out possible improvements. My obligations to previous writers are many, especially to De Morgan; at the same time, I am not aware of the existence of any work which occupies quite the same ground.

In conclusion, I return my sincere thanks to the many friends who have helped me with their sympathy and advice; and in particular to Professor A. Lodge, and Mr. G. J. Gibbs. To Mr. T. Wilson (Rothamstead Laboratory, Harpenden), Mr. A. E. Field (Grammar School, Bedford), and Mr. F. Taylor (Modern School, Bedford), who have undertaken the arduous task of working through the proof sheets, I owe more than I can express.

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CONTENTS.

ON COMPUTATION.

I. QUICKNESS IN PERFORMING AND VERIFYING ELEMENTARY WORK.

Addition.

1.

5
9
6
8
1
4
2
—
35

In adding up a column of digits, such as the one given on the left of this page, it is customary to do it either mentally or with actual vocal accompaniment as follows :—

Two and four, six ; and one, seven ; and eight, fifteen ; and six, twenty-one ; and nine, thirty ; and five, thirty-five. For rapidity it is not necessary to abandon the vocal statement of the result (though for other reasons it may be well to learn gradually to do without it) ; but supposing, for the present, that it is retained, the work should be accomplished with a *complete suppression of the "and," and also of the mention of the digits which have to be added.* Thus the student should simply say :—

Six, seven, fifteen, twenty-one, thirty, thirty-five,

laying emphasis on *thirty* and writing down *5* just as he says the word "*five*".

4937685
43299
376
8142488
976251
4397164
5732
18502995

Thus, in effecting the summation here given, the only numbers mentioned should be as follows :—

$6, 7, 15, 21, 30, 3'\underline{5}$
$6, 12, 17, 25, 32, 41, 4'\underline{9}$
$11, 12, 14, 18, 21, 23, 2'\underline{9}$
$7, 14, 20, 22, 25, 3'\underline{2}$
$12, 19, 23, 27, 3'\underline{0}$
$6, 15, 16, 2'\underline{5}$
$6, 14, 1'\underline{8}$

the digits underlined being written down just when their names are pronounced, and emphasis being laid on the words or syllables corresponding to those accented, for the sake of enabling the memory to retain the number to be carried.

As a check add *downwards;* this is preferable to doing the same work over a second time, as a mistake once made is likely to be repeated. The student should aim at *rapidity* from the first, and attain *correctness* by continual practice : not "slow and sure," but "quick and sure" must be his motto.

2. Facility should be obtained in adding numbers written in line across the page (*i.e.*, in doing *cross tots*) as well as those arranged in column down the page as in the example given above. If the total is required on the right-hand side, add from left to right, writing down the successive digits, as before, at the time of saying them; as a check add from right to left, and *vice versâ.*

If, however, the arrangement of the numbers to be added is left to the computer, he should almost always write them *in column* in preference to writing them *in line*, especially when the numbers are of more than four digits.

Some who read this may think the arrangement recommended so obviously dictated by common-sense that any remarks on the matter are superfluous. Long experience has, however, convinced the author of the necessity of insisting on it. In the addition of logarithms, for instance, beginners seem frequently to take it for granted that the proper way is to write them across the paper; teachers should take especial care to prevent the formation of this bad habit, and to compel its disuse if it has been already formed.

3. The student who requires numerous ready-made examples for practice in addition would do well to purchase *Civil Service Tots*, 1s. (Longmans).

Teachers who wish to drill their pupils in addition will find it a good plan to cyclostyle sheets of tots to be given to their classes. Sheets of ordinary and cross tots may be easily made, for individual or class use, by writing down sets of numbers in the way given below, leaving blanks for the totals :—

85672	74923	64321	
98325	84651	67896	
46458	73287	45321	
			640854

Here the same total 640854 of all the nine numbers written down should be obtained :—

(1) by first adding *in column* and then adding the three totals *from left to right ;*

(2) by first adding *from left to right* and then adding the three totals *in column.*

Useful examples of addition are afforded by the construction of Tables of Multiples.

Ex. To form a table of multiples of 369528 up to nine times by repeated addition.

(1) Having commenced the table by writing down 369528, copy down the same number 369528 on a moveable slip of paper or card, thus $\boxed{369528}$

(2) Putting the slip just above the number commencing the table, thus $\boxed{369528}$

1. 369528

(3) Add the two numbers, forming twice 369528.

(4) Move the slip down one line and add again, forming thrice 369528.

(5) Continue this process, moving down the slip and adding alternately.

(6) Having obtained nine times 369528, to check the table repeat the process once more; the result should of course be 3695280.

SUBTRACTION.

4. The ordinary "school" method should be at once discarded in favour of the "shop" or "complementary" method; the problem in subtraction being looked upon as *an inverse question in addition.*

Thus if we are asked to find the remainder when 4789 is taken from 8473 we should state the question to ourselves mentally in the form "what number added to 4789 would make the sum 8473?" and do our work from this point of view, thus:—

8473	9 and $\underline{4}$ make 1′3, carry 1
4789	9 and $\underline{8}$ make 1′7, carry 1
3684	8 and $\underline{6}$ make 1′4, carry 1
	5 and $\underline{3}$ make 8

setting down the digits underlined just as we say them, and emphasising the "teen," as indicated by the accent, to prevent our forgetting to carry the 1.

Note that in working the examples we have added *downwards;* consequently, we check by adding upwards; 4, 1′3; 9, 1′7; 7, 1′4; 4, 8.

The student should be able to work with the minuend either above the subtrahend (as it is usually written) or below it.

5. The chief object of getting thoroughly used to the above method of doing subtraction is to be able to extend it subsequently to Long Division. It will serve as an introduction to this extension if the student works a few examples like the following *in one operation.*

(1) *From* 9786453 *take six times* 1432985.

Work thus :—

	6 times 5, 30 ; and $\underline{3}$; 3′3.
9786453	6 times 8, 48 ; and 3′ ; 51 and $\underline{4}$; 5′5.
1432985	6 times 9, 54 ; and 5′ ; 59 and $\underline{5}$; 6′4.
———	6 times 2, 12 ; and 5′ ; 17 and $\underline{9}$; 2′6.
1188543	6 times 3, 18 ; and 2′ ; 20 and $\underline{8}$; 2′8.
	6 times 4, 24 ; and 2′ ; 26 and $\underline{1}$; 2′7.
	6 times 1, 6 ; and 2′ ; 8 and $\underline{1}$; 9.

(2) *From* 6485324 *take away the sum of* 57364, 485972, 2387542, *and* 396485.

	Work thus :—
6485324	5, 7, 9, 13 and $\underline{1}$; 1′4.
57364	9, 13, 20, 26 and $\underline{6}$; 3′2.
485972	7, 12, 21, 24 and $\underline{9}$; 3′3.
2387542	9, 16, 21, 28 and $\underline{7}$; 3′5.
396485	12, 20, 28, 33 and $\underline{5}$; 3′8.
3157961	6, 9, 13 and $\underline{1}$; 1′4.
	3 and $\underline{3}$; 6.

The method used in (2) may sometimes be applied with advantage in Practice in conjunction with the method explained in Pendlebury's *Arithmetic*, p. 197 (v.), p. 200 (iv.), and in Multiplication.

(i) Find the value of $512\frac{2}{3}$ things at 16s. $4\frac{3}{4}$d. each.

Here £1 − 16s. $4\frac{3}{4}$d. = 3s. $7\frac{1}{4}$d.

3s. 4d.	£$\frac{1}{6}$	512·6667
2d.	$\frac{1}{20}$	85·4444
1d.	$\frac{1}{2}$	4·2722
$\frac{1}{4}$d.	$\frac{1}{4}$	2·1361
		·5340
		£420·2800
		5·6
		7·2

Result, £420 5s. 7d.

(ii) Multiply 127·4 by ·9889.
Here ·9889 = 1 − ·0111.

$$\begin{array}{r} 127\cdot4 \\ \hline 1\cdot274 \\ \cdot1274 \\ \cdot01274 \\ \hline 125\cdot98586 \end{array}$$

For a further example of the same method in connection with contracted multiplication, see p. 15.

6. One special piece of subtraction which has often to be done is to obtain the *Arithmetical Complement* of a given number; *i.e.*, to find what must be added to the given number to make up the next higher power of 10.

Thus, by the Arithmetical Complement of 47365281 is meant the number that must be added to it to make up 100000000.

The rule is easily seen to be: *Proceeding from left to right, write under each digit but the last its defect from* 9; *under the last write its defect from* 10.

Thus:—given number, 47365281.
arithmetical complement, 52634719.

Similarly the arithmetical complements of

	436	28	9	2·3	·72614	
are	564	72	1	7·7	·27386	respectively.

In the example worked on p. 5, 3s. 7¼d. may be looked upon as a sort of *arithmetical complement* of 16s. 4¾d.

Note that the arithmetical complement of the sum of several numbers can be found, by the method on p. 5, without finding the actual sum.

Ex. (1) From 875231984 take 6 times 137984263
(2) From 940375864 take 5 times 98640722
(3) From 685732981 take 12 times 4793685

(4) From

	(i)	(ii)
From	85674323	73258694732
Take the sum of	598476	54863704823
	2485723	87643216
	54967219	8749438762
	880326	348796475
	9532178	2370589
	86421	
	378	

(5) Find the arithmetical complement of the sum of 46731, 52840, 64379, 1985673, 14950, without finding the sum itself.

(6) Find by Practice the value of 56302 articles at £5 18s. $3\frac{1}{2}$d. each.

Use the method of p. 5, taking parts only for 1s. $8\frac{1}{2}$d., the "complement" of 18s. $3\frac{1}{2}$d.

7. Self-verifying examples in addition and subtraction (depending on the identity $\overline{a - b} + \overline{b - c} + \overline{c - d} = a - d$) can be formed for practice of any required length, thus:—

319563124	85647293	70981764	319563124
85647293	70981764	25899973	25899973
233915831	14665529	45081791	293663151

Here the successive differences found by working the first three columns give, when added from left to right, the difference 293663151 of the first and last of the four numbers chosen. The same principle may also be utilised thus:—

$$
\begin{array}{rrcrcr}
(7) & 917469321 & - & 38300642 & = & 879168679 \\
 & 38300642 & - & 297461 & = & 38003181 \\
 & 297461 & - & 5993 & = & 291468 \\
\hline
 & 917469321 & - & 5993 & = & 917463328
\end{array}
$$

Here the subtraction is done across and the addition in column.

Only a small number of examples are given below on addition and subtraction; since private students who wish for practice can make up as many examples as they think desirable, and teachers who wish to drill their pupils can easily prepare, by cyclostyle or chromograph, *sheets* of such examples with vacant spaces for the answers.

	(8)		(9)		(10)		(11)
From	8267314957		659872143		493126459		826731495
	659872143		493126459		205126301		205126301
	____	+	____	+	____	=	____

(12) 7239587192005 − 5964720864229 =
(13) 5964720864229 − 4999888456301 =
(14) 4999888456301 − 2781130277568 =
(15) ____ − ____ = ____

Multiplication.

8. Multiply by the digits of the multiplier successively exactly in the reverse order to that usually taught; *i.e.*, begin with the digit of highest order, and end with that in the units place. Thus, in multiplying 52367 by 2459 take the digits in order from left to right, 2, 4, 5, 9, and arrange the work thus:—

```
   52367
    2459
 -------
104734
 209468
  261835
   471303
---------
128770453
```

The advantage of this method will become more apparent when the student is learning to *abridge* his work for *approximate* results.

To verify the work use the old method of *casting out the nines*, as follows :—

The remainders, when 52367 and 2459 are divided by 9, are respectively 5 and 2.

The remainder, when the product 10 of these two remainders is divided by 9, is 1 ; this, if the work is correctly done, will be the remainder when 128770453 is divided by 9, and on trial this is found to be the case.

Any other number besides 9 might be "cast out" in a similar way, and the result ought always to stand the test.

But the chance of undetected error in the result would be greater. Casting out *twos* would hardly verify more than the units digit.

9 is chosen because we can find the remainders most readily ; the remainder left when any number is divided by 9 being the same as that left when the sum of its digits is divided by 9. Thus, taking the numbers 52367, 2459, 128770453, the sums of the digits are respectively 23, 20, 37, and the remainders consequently 5, 2, 1.

9. The student who is acquainted with Algebraical Subtraction may almost as easily "cast out" elevens as nines by the following rule, taken from an appendix to the fifth edition of De Morgan's *Arithmetic* :—

Subtract the first figure from the second, the result from the third, the result from the fourth, and so on. The final result, or the rest of eleven, if the figure be negative, is the remainder required.

For the number 52367 we have 5 from 2, -3 ; -3 from 3, 6 ; 6 from 6, 0 ; 0 from 7, $\underline{7}$.

In practice, when used to the method, we should simply say -3, 6, 0, $\underline{7}$.

For 2459 we have 2, 3, $\underline{6}$.
For 128770453 we have 1, 7, 0, 7 − 7, 11 − 6, $\underline{9}$.
Now $\underline{9}$ is also the remainder when 7 × 6 is divided by 11.

10. In multiplying decimals, do not, as it is usual to do, omit the points from the factors and afterwards point off the product, but write the factors down so that point comes under point, and let each partial product be pointed; this only involves pointing the first partial product correctly, the rest following naturally. Take as an example the product

(i) of 2457·086 and 328·19;
(ii) of 24570·86 and ·032819.

```
  2457·086
   328·19
----------
737125·8
 49141·72
 19656·688
   245·7086
   221·13774
------------
806391·05434
```

Here, beginning to multiply by the 3 and saying "3 times 6, 18" we write the 8 not under the 6 from which it is derived, but *two places further to the left*, because the 3 is *two places to the left of the units place.*

```
24570·86
     ·032819
-------------
  737·1258
   49·14172
   19·656688
     ·2457086
     ·22113774
-------------
  806·39105434
```

Here the 8 is written *two places further to the right* than the 6 from which it is derived, because the 3 is *two places to the right of the units place.*

Ex. (16) Multiply 879654 by 79863.
Ex. (17) Multiply 987654321 by 123456789.
Ex. (18) Multiply 530674 by 45007.

Ex. (19) Find the square of 886437165.
Ex. (20) Find the cube of 1589.

11. The use of an extended multiplication table will sometimes expedite work; thus, if we wish to find 52 times 3141592 we may extract 52 times 31, 52 times 41, 52 times 59, 52 times 2 from Destéract's *Table de Pythagore* (extending to 99 times 99 and costing only one franc), and obtain the product by addition, thus:—

```
  3141592
  -------
1612
  2132
    3068
       104
---------
163362784
=========
```

12. When the multiplier consists of only two digits the work should be shortened by adding the result of the multiplication of the second digit to that already obtained. Thus in multiplying 52367 by 24, after the multiplication by 2 we should proceed as indicated by the figures.

```
                        2'8
  52367      24, 26, 3'0
     24      12, 15, 1'8
 ------       8,  9, 1'6
 104734      20, 21, 2'5
1256808           2,  2
=======               1
```

This method can be easily extended to a multiplier with any number of digits. Thus, to return to the example given on p. 8, the work might be done in the way indicated below:—

$$\begin{array}{r}
52367 \\
2459 \\
\hline
104734 \\
1256808 \\
12829915 \\
\underline{128770453} \\
\hline\hline
\end{array}$$

the second, third and fourth results being respectively 24 times, 245 times, and 2459 times 52367.

It should be noticed that to multiply by a number of two digits of which one is unity, only requires one line of work; the steps being indicated thus:—

$$\begin{array}{r}
52367 \\
14 \\
\hline
\underline{733138} \\
\hline\hline
\end{array}
\qquad\qquad
\begin{array}{r}
2'\underline{8} \\
24,\ 26,\ 3'\underline{3} \\
12,\ 15,\ 2'\underline{1} \\
8,\ 10,\ 1'\underline{3} \\
20,\ 21,\ 2'\underline{3} \\
\underline{7}
\end{array}$$

When first using this method the student is apt to omit the last step. (Here $2 + 5 = 7$.)

Excellent examples in combined multiplication and addition can be found in reduction and in the evaluation of algebraical expressions.

Ex. To reduce £5 3s. $7\frac{1}{4}$d. to farthings.

£	s.	d.	
5	3	$7\frac{1}{4}$	
	103	1243	
		4973	farthings.

Ex. To find the value of $3x^3 + 7x^2 + 5x + 4$ when $x = 4$.

$$\begin{array}{cccc}
3x^3 + & 7x^2 + & 5x + & 4 \\
& 19 & 81 & 328.
\end{array}$$

Here the work is exactly analogous to the reduction process of the previous example. ($x^3 = 4x^2$, $x^2 = 4x$, $x = 4$ units.)

$$19 = 4 \times 3 + 7.$$
$$81 = 4 \times 19 + 5.$$
$$328 = 4 \times 81 + 4.$$

Ex. (21) Evaluate the same expression when $x = 5$.

13. It is unnecessary to write down the algebraical denominations. Thus, in evaluating the same expression when $x = \cdot 5$, we need not write down more than this :—

3	7	5	4
	8·5	9·25	8·625.

Ex. (22) Evaluate the above expression when $x = \cdot 4$.

If any one of the denominations is missing, we must fill the place of its coefficient by a 0 ; thus :—

Ex. To find the value of $3x^5 + 4x^2 + 7x + 5$ when $x = 2$.

3	0	0	4	7	5
	6	12	28	63	<u>131</u>.

Ex. (23) Evaluate $3x^5 + 4x^3 + 7x + 5$ (i) when $x = 2$, (ii) when $x = \cdot 5$.

The following is an example of the multiplication of decimals ($0 \cdot 4571 \times 2 \cdot 856$) :—

$$\begin{array}{ll}
\cdot 04571 & \\
\underline{2 \cdot 856} & \\
\cdot 09142 & (= \cdot 04571 \times 2) \\
\cdot 127988 & (= \cdot 04571 \times 2 \cdot 8) \\
\cdot 1302735 & (= \cdot 04571 \times 2 \cdot 85) \\
\cdot 13054776 & \;\; = \cdot 04571 \times 2 \cdot 856.
\end{array}$$

14. When the same multiplier (such as π or the modulus μ) has frequently to be used, work may be saved by the formation of a table of its multiples. Thus, an electrician who had frequently to convert "legal ohms" into "B.A. units"

and *vice versâ* might shorten his labours by the use of the following tables :—

Ohms	1	1·0112	B.A. units	1	0·9889
	2	2·0224		2	1·9778
	3	3·0336		3	2·9667
	4	4·0448		4	3·9556
	5	5·0560		5	4·9445
	6	6·0672		6	5·9334
	7	7·0784		7	6·9223
	8	8·0896		8	7·9112
	9	9·1008		9	8·9001

$$\text{Thus } 53{\cdot}8 \text{ ohms} = \left.\begin{array}{r} 50{\cdot}56 \\ +\ 3{\cdot}0336 \\ +\ {\cdot}80896 \end{array}\right\} \text{B.A. units,}$$

$$\text{and } 53{\cdot}8 \text{ B.A. units} = \left.\begin{array}{r} 49{\cdot}445 \\ +\ 2{\cdot}9667 \\ +\ {\cdot}79112 \end{array}\right\} \text{ohms.}$$

Examples.

Ex. (24) Obtain the values of $5x^3 - 7x^2 + 8x - 9$ (i) when $x = 3$; (ii) when $x = 5$; (iii) when $x = 10$.

Ex. (25) Obtain the value of $4x^3 + 5x^2 + 7x + 3$ (i) when $x = \frac{1}{2}$; (ii) when $x = \frac{1}{4}$; (iii) when $x = {\cdot}1$.

Ex. (26) Find the number of shillings, pence, and farthings successively in £19 17s. 4¾d.

Ex. (27) Obtain the successive products of ·03659 by 2·8, 2·83, 2·835. (See p. 13, Ex.)

Ex. (28) Make a table of multiples of 123456789, and use it to obtain the product of that number (i) by 907050301, and (ii) 80604020; (iii) by 648275; (iv) by 60408020705.

15. A method of obtaining products by finding the difference of two numbers extracted from a "quarter-square" table depends on the following algebraical identity :—

$$\tfrac{1}{4}(a + b)^2 - \tfrac{1}{4}(a - b)^2 = ab.$$

Thus, to find 573 × 386.

$$573 + 386 = 959 \text{ whose } \tfrac{1}{4} \text{ sq.} = 229920\tfrac{1}{4}.$$
$$573 - 386 = 187 \text{ whose } \tfrac{1}{4} \text{ sq.} = 8742\tfrac{1}{4}.$$
$$573 \times 386 = 221178.$$

Of course a table of squares may be utilised for this method if a table of quarter squares is not available. A table of quarter squares of nos. from 1 to 5100 is given at the end of the new edition of Chambers' *Mathematical Tables.*

16. Special methods, depending on the form of the multiplier, may sometimes be devised for obtaining a product. A few examples are given below.

(i) Multiply 5674 by 999.

Since 999 = 1000 − 1, we obtain the result by the following piece of subtraction :—

$$\begin{array}{r} 5674000 \\ 5674 \\ \hline 5668326 \end{array}$$

(ii) Multiply 5642 by 9997.

We subtract three times 5642 from 56420000, doing the multiplication and subtraction concurrently as on p. 5.

$$\begin{array}{r} 56420000 \\ \hline \underline{\underline{56403074}} \end{array}$$

(iii) Multiply 578·643 by 2·987.

Here 2·987 = 3 − ·013.

$$\begin{array}{r} 578{\cdot}643 \\ \hline 1735{\cdot}929 \\ \hline 5{\cdot}78643 \\ 1{\cdot}735929 \\ \hline 1728{\cdot}406641 \end{array}$$

Compare the example worked on p. 6, *where the same method is used.*

(iv) Multiply 89763 by 25.

$$89763 \times 25 = \frac{8976300}{4}$$
$$= 2244075.$$

Similarly for multiplication by 2·5, ·25, ·025, etc.

(v) Multiply 89763 by 125.

$$89763 \times 125 = \frac{89763000}{8}$$
$$= 11220375.$$

Similarly for multiplication by 12·5, 1·25, ·125, etc.

Ex. (29) Multiply 69547832 (i) by 999 × 999, (ii) by 99 × 99 × 99, (iii) by 98 × 98 × 98.

Ex. (30) Multiply 6977662315 (i) by 97 × 98 × 99, (ii) by 997 × 998.

Ex. (31) Multiply 3·8467 by 48·97.

Use the method of Ex. iii, p. 15, noting that

$$48{\cdot}97 = 50 - 1{\cdot}03.$$

Ex. (32) Multiply 98903628 by 9945 (*i.e.*, by 10000 − 55).

Ex. (33) Multiply 6·15365513 by 267 (*i.e.*, by 300 − 33).

Ex. (34) Find the product of 8736921054 when multiplied (i) by 25, (ii) by 25 × 125.

Ex. (35) Find (i) 67·326 × ·0125; (ii) 67·326 × ·25 × ·25.

Division.

17. If the student is not provided with a table of multiples of the divisor, he should adopt what is called the "Italian method," which consists in writing down the remainders while doing the multiplication, and omitting the multiples of the divisor, the remainders being obtained by the method of subtraction recommended on p. 4.

E.g., to divide 118603127 by 52739 we work thus, obtaining the first remainder 13125, as follows:—

	$\overline{2248}$	
52739)	118603127	Twice 9, 18, and $\underline{5}$, 2′3
	131251	Twice 3, 6, 8, and $\underline{2}$, 1′0
	257732	Twice 7, 14, 15, and $\underline{1}$, 1′6
	467767	Twice 2, 4, 5, and $\underline{3}$, 8
	45855	Twice 5, 10, and $\underline{1}$, 11

then bringing down the next figure, 1, and proceeding as before.

As in multiplication "casting out nines" may be used for a test of correctness.

As further examples we append the work:—

(i) For finding the G.C.M. of 6281 and 326041.

(ii) For finding the square root of 8139609.

					$\underline{2853}$
		$\underline{51,1,10}$		2	8,13,96,09
(i)	6281)	326041	(ii)	48	413
	$\underline{571}$	11991		565	2996
		5710		5703	17109
	G.C.M. = 571			Sq. rt. = 2853	

18. To extract the cube root of a number in the most concise and expeditious way, the student will have to use both—

(i) The combination of multiplication and addition explained on p. 11.

(ii) The combination of multiplication and subtraction (called "Italian") explained on p. 16, and just used, in the above example of the subtraction of a square root.

19. If a divisor is of frequent occurrence it may help the computer to keep a table of its multiples.*

* When pupils are beginning to learn Long Division the teacher should always insist on the formation of a table of multiples of the divisor.

Ex. (36). Divide 9144947333447645175 (i) by 493827165, (ii) by 2962962963.

Ex. (37) Divide 6878494263826549998 (i) by 87085746, (ii) by 8621488854, (iii.) by 8612867365146.

Ex. (38) Divide 7857708474932237225 (i) by 176287433, (ii) by 2659311495.

Ex. (39) Find the G.C.M. of 23632464 and 43800771, and reduce to lowest terms $\frac{23632464}{43800771}$.

Ex. (40) Find the square root and the fourth root of 490796923761.

Ex. (41) Divide 702·2992494 (i) by 7·9863, (ii) by ·043969.

Ex. (42) Find the G.C.M. of 180·628 and 29·0658.

Ex. (43) Find the fourth root of ·0000531441.

20. When the Italian method has been thoroughly learned the computer may occasionally find an advantage in using it in combination with a way of writing down the successive remainders different to that usually employed. It is given in Lang's *Higher Arithmetic.** Thus, in the division of 1248631742953, given below, the successive remainders, 212, 54, 25, 251, 186, 51, 253, 208, 13, 133, are written diagonally instead of across the page, from left to right.

As we proceed, the work *stretches across the paper from left to right,* instead of *lengthening downwards.*

```
259 ) 1248631742953
         245 61383
        152 85501
       2   1 22
      ─────────────
         4820971980
```

The method allows the quotient to be written figure by figure beneath the rest of the work (*as in Short Division*), an

* A similar arrangement is described in O'Gorman's *Intuitive Calculations.*

arrangement of considerable advantage when several successive divisions have to be performed, especially when the divisors are short compared with the dividends.

The division of 43825761 by 19 is given below as a further example :—

```
19 ) 43825761
       51 137
        1 1
     --------
      2306619
```

Here the successive remainders are 5, 1, 12, 11, 3, 17, 0.

The method is specially applicable in finding Present Worth (see p. 157).

21. We may utilise the extended multiplication table (mentioned on p. 11) in performing division. In the following the partial products are taken, as before, from the *Table de Pythagore*.

```
3141592 ) 1656995001296 ( 527438
          1612
             2132
                3068
                   104
          -------------
             233671612
             ---------
             2294
                3034
                   4366
                      148
             ------------
                119380496
                ---------
                1178
                   1558
                      2242
                         76
                ---------
                . . . . . . .
```

Here the remainder 2336716 is found by adding up the partial products taken from the multiplication table, and simultaneously subtracting their sum from 165699500 after the manner of the example worked on p. 6. Similarly the remainder 1193804 is obtained by adding up the partial products and subtracting simultaneously from 233671612.

This method is not recommended as a substitute for the Italian, but it may be employed as *a useful check on a result obtained by the former.*

22. Special methods, depending on the form of the divisor, may sometimes be devised for obtaining a quotient; a few examples are given below:—

(i) Divide 45326107 by 999.

Since 999 = 1000 − 1 we obtain the result by the following piece of addition:—

$$\begin{array}{r} 45326|107 \\ 45|326 \\ |45 \\ \hline 45371|478 \end{array}$$

the quotient being 45371 and the remainder 478.

If a digit has to be carried from right to left of the vertical line the same digit must be added to the number on the right of it.

Thus, 873635421 ÷ 999 gives quotient 874509 with remainder 930 since unity is carried.

$$\begin{array}{r} 873635|421 \\ 873|635 \\ |873 \\ \hline 874509|929 \\ \hline \end{array}$$

(ii) Divide 5674 by 25.

$$\frac{5674}{25} = \frac{5674 \times 4}{100} = 226{\cdot}96$$

(iii) Divide 5674 by 125.

$$\frac{5674}{125} = \frac{5674 \times 8}{1000} = 45{\cdot}392$$

Ex. (44) Divide (i) 844917237 by 999, (ii) 732481985169 by 9999, (iii) 8289323163 by 99 times 99.

Ex. (45) Divide (i) 1459315425 by 25, (ii) 76328425 by ·25, (iii) 729657·7125 by 125.

Vulgar Fractions.

23. Vulgar fractions of an extremely complex character, such as :—

$$\frac{\dfrac{\frac{1}{3}-\frac{1}{5}}{1+\frac{1}{15}} + \dfrac{\frac{1}{5}-\frac{1}{7}}{1+\frac{1}{35}}}{1 - \dfrac{\frac{1}{3}-\frac{1}{5}}{1+\frac{1}{15}} \text{ of } \dfrac{\frac{1}{5}-\frac{1}{7}}{1+\frac{1}{35}}}$$

do not often occur in calculations, though they afford useful exercises in neatness and dexterity; they belong, in fact, to Arithmetical Gymnastics. The following remarks apply to such as are likely to occur in practical work.

24. In simplifying complex fractions students should apply the theorem :—

"*The value of a fraction is unaltered by multiplying both numerator and denominator by the same number*" much more directly than they usually venture to do: as directly, in fact, as they apply the converse principle as to dividing both numerator and denominator when they are reducing fractions to their lowest terms.

Thus the complex fraction $\dfrac{2\frac{3}{4}}{5\frac{1}{8}}$ should be reduced to the simple equivalent fraction $\frac{22}{41}$ at once by multiplying numerator and denominator by 8, without any recourse to the process: $\dfrac{2\frac{3}{4}}{5\frac{1}{8}} = \frac{11}{4} \div \frac{41}{8} = \frac{11}{4} \times \frac{8}{41} = \frac{22}{41}$.

As another example: $\dfrac{\frac{1}{3} - \frac{1}{5}}{1 + \frac{1}{15}} = \dfrac{5 - 3}{15 + 1} = \dfrac{2}{16} = \dfrac{1}{8}.$

The same method frequently applies to fractions in which the numerator and denominator are concrete quantities. Thus :—

$$\frac{\pounds 4\ 17\text{s. } 3\text{d.}}{\pounds 5} = \frac{\pounds 19\ 9\text{s.}}{\pounds 20} = \frac{389}{400}.$$

25. In reducing fractions to their lowest terms it is well to avoid, if possible, the tedious process of finding the G.C.M. of the numerator and denominator; and the rules for finding whether a given number is divisible by 9 or by 11 (explained under the head of "casting out" nines or elevens) are often useful.

In trying to find out whether a given number has factors or not (or whether it is *composite* or *prime*) it should be borne in mind (*a*) that only prime divisors need be tried; (*b*) that we need not try any divisor greater than the square root of the given number.

Thus in trying to find the factors of 331 the prime numbers which are less than the square root are 2, 3, 5, 7, 11, 13, 17. Of these, 2 and 5 may be rejected at once, and 3 and 11 may be tried by the "casting out" process; leaving 7, 13, 17 to be tried by ordinary division: if all these fail we rightly judge 331 to be a prime.

A table of prime and composite odd numbers under 10000, giving in the case of a composite its *lowest prime factor*, is given in Hutton's *Mathematical Dictionary* under Primes.

Ex. (46) Simplify in the way explained in art. 24 the following fractions :—

(i) $\dfrac{2\frac{1}{3}}{3}$, (ii) $\dfrac{5}{8\frac{1}{9}}$, (iii) $\dfrac{2\frac{1}{3}}{3\frac{1}{2}}$, (iv) $\dfrac{\pounds 3\ 5\text{s. } 7\frac{1}{2}\text{d.}}{\pounds 4\ 8\text{s. } 4\frac{1}{4}\text{d.}}$

(v) $\dfrac{2 \text{ tons } 11 \text{ cwts. } 3 \text{ qrs.}}{5 \text{ tons } 13 \text{ cwts. } 2 \text{ qrs.}}$

Ex. (47) Find the prime factors of (i) 40320, (ii) 399168, (iii) 4790016, (iv) 62044840173323943936.

Ex. (48) Reduce to their lowest terms:—

$$\text{(i)}\ \frac{558558}{791406}, \qquad \text{(ii)}\ \frac{494505}{1285713}.$$

26. It is usual in school text-books and in school work, to keep vulgar fractions and decimal fractions much too distinct from one another. Each kind has its own special advantage, and it is frequently useful to avail oneself of both *in the same piece of work.* Thus there is no objection to the mixed notation illustrated in the following statements: $\frac{1}{19} = \cdot05\frac{5}{19}$; $\frac{1}{24} = \cdot04\frac{1}{6}$.

Again, in finding the product of two numbers, one of which is expressed as a decimal and the other as a vulgar fraction, it is not only unnecessary, but frequently very inadvisable to do what the ordinary school-boy would be almost certain to do, *i.e.*, express each in the same way before multiplying. For instance, if we have to find the product of 4·7892 and $6\frac{1}{3}$ it is best to arrange the work thus:—

$$\begin{array}{r} 4\cdot7892 \\ \hline 28\cdot7352 \\ 1\cdot5964 \\ \hline 30\cdot3316 \end{array}$$

This example illustrates the general principle that it is advantageous to use *decimal fractions to operate on*, but *vulgar fractions to operate with.*

Ex. (49) Multiply 1 − ·0004 by $3\frac{1}{7}$.

Special devices in connection with operations like the above may occasionally be adopted with advantage. For instance, if we had to multiply 5·89673 by $4\frac{7}{8}$ it would save trouble to use the identity $4\frac{7}{8} = 5 - \frac{1}{8}$, thus:—

$$\begin{array}{r} 5\cdot89673 \\ \hline 29\cdot48365 \\ \cdot73709125 \\ \hline 28\cdot74655875 \end{array}$$

Sometimes the methods of "Practice" may be usefully employed. Take for examples the product of (i) 43·576 and $5\frac{9}{16}$, (ii) ·02739 and $\frac{13}{32}$.

$$\begin{array}{lr}
\text{(i)}\ \frac{8}{16}\ \frac{1}{2} & \underline{43{\cdot}576} \\
 & 217{\cdot}880 \\
\frac{1}{16}\ \frac{1}{8} & 21{\cdot}788 \\
 & \underline{2{\cdot}7235} \\
 & 242{\cdot}3915
\end{array}
\qquad
\begin{array}{lr}
\text{(ii)}\ \frac{8}{32}\ \frac{1}{4} & \underline{{\cdot}02739} \\
\frac{4}{32}\ \frac{1}{2} & {\cdot}0068475 \\
\frac{1}{32}\ \frac{1}{4} & {\cdot}00342375 \\
 & \underline{{\cdot}0008559375} \\
 & {\cdot}0111271875
\end{array}$$

Ex. (50) Find the products of (i) 523·845 and $6\frac{7}{18}$, (ii) ·0004582 and $\frac{19}{64}$.

Further illustrations of the above methods will be found on pp. 80, 81.

The student's attention is specially directed to the *principle* of "Practice," viz., *the decomposition of a fraction or set of fractions into a series of other fractions, each of which has unity for its numerator.* In ordinary text-books the method is only applied to fractions of *concrete* quantities. Thus when we take "aliquot parts" for 16s. $7\frac{1}{2}$d. we replace the set of fractions $\frac{16}{20} + \frac{7\frac{1}{2}}{240}$ by the equivalent set $\frac{1}{2} + \frac{1}{4} + \frac{1}{20} + \frac{1}{40} + \frac{1}{160}$; but the author has been accustomed for some time to urge its use for *abstract* quantities as well, and was glad to find himself supported by Prof. A. Lodge in his article on "Reductions and Approximations" in the *Mathematical Gazette* for April, 1894. The following decompositions may be found useful in themselves, and may also serve to suggest others:—

$$\tfrac{2}{15} = \tfrac{1}{10} + \tfrac{1}{30};\ \tfrac{18}{25} = \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{50}$$
$$\tfrac{5}{27} = \tfrac{1}{6} + \tfrac{1}{54};\ \tfrac{9}{32} = \tfrac{1}{4} + \tfrac{1}{32}$$
$$\tfrac{44}{63} = \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{42} + \tfrac{1}{126};\ \tfrac{2}{55} = \tfrac{1}{30} + \tfrac{1}{330}$$
$$\tfrac{7}{22} = \tfrac{1}{6} + \tfrac{1}{11} + \tfrac{1}{22} + \tfrac{1}{66}\,*$$

* It seems that the ancient Egyptians used such decompositions largely. In fact, the above were effected by an Egyptian computer, Ahmes, and are taken from an article (*Studi intorno alla Logistica Greco-Egiziana*) contributed by Prof. G. Loria to the *Giornale di Mathematiche* (vol. xxxii.), which deals with the arithmetic of the ancient Egyptians.

Though, as Prof. Loria shows, such decompositions can be effected for all fractions they are chiefly useful to the calculator when the denominator is (1) a power of some number less than 12; (2) the product of two numbers less than 12.

Occasionally differences of such fundamental fractions may be used instead of sums; thus: $\frac{13}{55} = \frac{1}{4} - (\frac{1}{110} + \frac{1}{220})$.

Involution.

27. If it is proposed to find the square, the cube, or the ourth or any higher power of a given number, the question may of course be treated as one of ordinary multiplication only, and a few questions in involution have been given to be worked as examples of that process. But there exists a special method, known as Horner's, of performing Involution, which not only is more expeditious than ordinary multiplication for its direct object, but leads easily to the only comprehensive and expeditious method of performing the inverse process of Evolution. The student acquainted with Algebra already knows, or may easily verify by multiplication, the identity $(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$. His attention is drawn to the following method of obtaining the right-hand expression by the series of operations indicated in the system of two columns given below, in which $(a + b)^2$, and $(a + b + c)^2$ are successively formed by additions and multiplications from a^2:—

$$\begin{array}{ll}
a & a^2 \\
\underline{a} & \underline{\;\;2ab + b^2} \\
2a + b & a^2 + 2ab + b^2 \\
\underline{\;b} & \underline{\;+ 2ac + 2bc + c^2} \\
2a + 2b + c & a^2 + 2ab + b^2 + 2ac + 2bc + c^2
\end{array}$$

Here the left column is formed by repeated additions only, the right by additions and multiplications.

$2ab + b^2$ being obtained by multiplying the $2a + b$ of the

left column by b, and the $2ac + 2bc + c^2$ of the right similarly obtained by multiplying $2a + 2b + c$ by c.

The squares of $a + b + c + d$, $a + b + c + d + e$, etc., could be found by an extension of the above columns.

28. The student is strongly recommended to use the above method for obtaining the squares of both Algebraical and Arithmetical quantities, *compressing and contracting as advised in the sections on the simpler operations* (see pp. 12, 13). As examples of the process recommended the squares of $x^3 - 4x^2 + 5x - 3$ and 83792 are worked out, addition and multiplication being combined, as on pp. 12, 13, and some unnecessary cyphers omitted.

(i)

x^3	x^6
$2x^3 - 4x^2$	$x^6 - 8x^5 + 16x^4$
$2x^3 - 8x^2 + 5x$	$x^6 - 8x^5 + 26x^4 - 40x^3 + 25x^2$
$2x^3 - 8x^2 + 10x - 3$	$x^6 - 8x^5 + 26x^4 - 46x^3 + 49x^2 - 30x + 9$

Here from x^6 (the square of x^3) we form in succession the squares of $x^3 - 4x^2$, $x^3 - 4x^2 + 5x$, and $x^3 - 4x^2 + 5x - 3$.

The result might be checked by forming the square of $3 - 5x + 4x^2 - x^3$ by the same method.

(ii)

8	6400
163	688900
1667	70056900
16749	7020764100
167582	7021099264

Here from 6400000000 (the square of 80000) we form in succession the squares of 83000, 83700, 83790, and 83792. Useless cyphers are omitted, two only being retained in each case to mark the position of the figures in the next line.

Ex. (51) Find in succession the squares of $x^3 - 5x^2$, $x^3 - 5x^2 + 4x$, $x^3 - 5x^2 + 4x - 3$.

Check the final result obtained by squaring $3 - 4x + 5x^2 - x^3$.

Ex. (52) Find by the above process the square of $x^3 + 3x^2 + 4x + 5$.

Ex. (53) i. Find, by the process used in ii., the square of 83793; ii. find also the squares of 83782 and of 84792.

29. The method may be used to extend a table of squares. Thus, being given that $(987)^2 = 974169$, we can find the square of 987·5.

$$\begin{array}{lcl}
987 & \qquad\qquad & 974169 \\
1974{\cdot}5 & & 975156{\cdot}25
\end{array}$$

Ex. (54) Given $(986)^2 = 972196$, find by the above method the square of 986·5.

Ex. (55) Find the squares of 987·53 and of 987·534.

30. It is easy to extend the method to the evaluation of such algebraical expressions as $x^2 + 3x + 8$, $2x^2 + 5x + 11$, $3x^2 + 87x + 514$, etc., when x is a number of several digits. A careful examination of the following schemes of work will, it is believed, help the student more than a set of general directions.

(i) Evaluation of $x^2 + 3x + 8$ when $x = 567$.

$$\begin{array}{rrr}
1 & 3 & 8 \\
 & \underline{500} & \underline{251500} \\
 & 503 & 251508 \\
 & \underline{500} & \underline{63780} \\
 & 1003 & 315288 \\
 & \underline{60} & \underline{7910} \\
 & 1063 & 323198 \\
 & \underline{60} & \\
 & 1123 & \\
 & \underline{7} & \\
 & 1130 &
\end{array}$$

Here, from 251508 (the value of $x^2 + 3x + 8$ when $x = 500$) we form in succession the values 315288, 323198 of the same expression when $x = 560$, and when $x = 567$ respectively. The same work, contracted, would stand thus:—

1	3	8
	503	251508
	1003	315288
	1063	323198
	1123	
	1130	

Ex. (56) Evaluate $x^2 + 3x + 8$ when $x = 569$; with and without contraction of work.

Ex. (57) Evaluate $x^2 + 4x + 8$ when $x = 567$; contracting the work.

(ii) Evaluate $2x^2 + 5x + 11$ when $x = 567$.

2	5	11
	1000	502500
	1005	502511
	1000	127500
	2005	630011
	120	15813
	2125	645824
	120	
	2245	
	14	
	2259	

Here, from 502511 (the value of $2x^2 + 5x + 11$ when $x = 500$) we form in succession the values 630011, 645824 of the same expression when $x = 560$, and when $x = 567$ respectively.

The same work contracted would stand thus:—

2	5	11
	1005	502511
	2005	630011
	2125	645824
	2245	
	2259	

Ex. (58) Evaluate $2x^2 + 5x + 11$ when $x = 568$.

Ex. (59) Evaluate $2x^2 + 6x + 12$ when $x = 567$.

(iii) Evaluation of $3x^2 + 87x + 514$ when $x = 5672$ (work contracted).

3	87	514
	15087	75435514
	30087	94567714
	31887	96940504
	33687	97008730
	33897	
	34107	
	34113	

Ex. (60) Evaluate $3x^2 + 87x + 514$ when $x = 5671$.

Ex. (61) Evaluate $3x^2 + 87x + 514$ when $x = 567$.

Ex. (62) Evaluate $3x^2 + 87x + 514$ when $x = 56$.

Ex. (63) Evaluate $5x^2 + 97x + 513$ when $x = 5672$.

Ex. (64) Evaluate $5x^2 + 97x + 513$ when $x = 2345$.

(iv) Evaluation of $4617x^2 + 29318x$ when $x = 21{\cdot}75$.

4617	29318	
	121658	2433160
	213998	2651775
	218615	2810299·73
	223232	2821796·0625
	226463·9	
	229695·8	
	229926·65	

Ex. (65) Evaluate $4617x^2 + 29318x$ when $x = 31{\cdot}75$.

Ex. (66) Evaluate $5943x^2 + 54761x$ when $x = 1{\cdot}473$.

Ex. (67) Evaluate $561983x^2 + 28617x + 493$ when $x = 3{\cdot}4287$.

31. Proceeding to Cubes we start from the identities:—

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + \{3a^2 + (3a + b)b\}b$$
$$(a + b + c)^3 = (a + b)^3 + \{3\,(a + b)^2 + (3\,\overline{a + b} + c)c\}c,$$

and direct the student's attention to the following method of obtaining the right-hand expressions by the series of operations indicated in the system of three columns given on the opposite page, in which $(a + b)^3$ and $(a + b + c)^3$ are successively formed by additions and multiplications from a^3:—

Here the left column is formed by repeated additions only; the middle column is formed by the successive additions of the products $2a^2$, $3ab + b^2$, $3ab + 2b^2$, $3ac + 3bc + c^2$ obtained by multiplying the quantities $2a$, $3a + b$, $3a + 2b$, $3a + 3b + c$ in the left column by a, b, b and c respectively; the right column is found by the successive additions of the products $3a^2b + 3ab^2 + b^3$ and $3(a + b)^2c + 3(a + b)c^2 + c^3$ obtained by multiplying the quantities $3a^2 + 3ab + b^2$, $3(a + b)^2 + 3(a + b)c + c^2$ in the middle column by b and c respectively.

The student is strongly recommended to use the above method for obtaining the cubes of both Algebraical and Arithmetical quantities, *compressing and contracting as advised in the sections on the simpler operations* (see pp. 12, 13).

As examples of the process recommended, the cubes of (i) $x^3 - 4x^2 + 5x - 3$ and (ii) 83792 are worked out, addition and multiplication being combined as on pp. 12, 13, and some unnecessary cyphers omitted.

a	a^2	a^3
a	$2a^2$	$\underline{3a^2b + 3ab^2 + b^3}$
$\underline{2a}$	$\underline{3a^2}$	$(a + b)^3$
a	$3ab + b^2$	$\underline{3(a + b)^2c + 3(a + b)c^2 + c^3}$
$\underline{3a + b}$	$\underline{3a^2 + 3ab + b^2}$	$(a + b + c)^3$
$+ b$	$3ab + 2b^2$	
$\underline{3a + 2b}$	$\underline{3(a + b)^2}$	
b	$+ 3(a + b)c + c^2$	
$\underline{3(a + b) + c}$	$\underline{3(a + b)^2 + 3(a + b)c + c^2}$	

x^3	x^6	x^9
$2x^3$	$3x^6$	$x^9 - 12x^8 + 48x^7 - 64x^6$
$3x^3 - 4x^2$	$3x^6 - 12x^5 + 16x^4$	$x^9 - 12x^8 + 63x^7 - 184x^6 + 315x^5 - 300x^4 + 125x^3$
$3x^3 - 8x^2$	$3x^6 - 24x^5 + 48x^4$	$x^9 - 12x^8 + 63x^7 - 193x^6 + 387x^5 - 534x^4 + 512x^3 - 333x^2 + 135x - 27.$
$3x^3 - 12x^2 + 5x$	$3x^6 - 24x^5 + 63x^4 - 60x^3 + 25x^2$	
$3x^3 - 12x^2 + 10x$	$3x^6 - 24x^5 + 78x^4 - 120x^3 + 75x^2$	
$3x^3 - 12x^2 + 15x - 3$	$3x^6 - 24x^5 + 78x^4 - 129x^3 + 111x^2 - 45x + 9$	

Here from x^9 (the cube of x^3) we form in succession the cubes of $x^3 - 4x^2$, $x^3 - 4x^2 + 5x$ and $x^3 - 4x^2 + 5x - 3$.

The result might be checked by forming the cube of $-3 + 5x - 4x^2 + x^3$ by the same method.

(ii)	8	64	512000
	16	19200	571787000
	243	19929	586376253000
	246	2066700	588269823939000
	2497	2084179	588311949529088
	2504	210170700	
	25119	210396771	
	25128	21062292300	
	251372	21062795044	

Here from 512000000000000 (the cube of 80000) we form in succession the cubes of 83000, 83700, 83790, 83792. Useless cyphers are omitted from the work; two only in the middle column, and three only in the right column being retained in each case to mark the position of the figures in the next line.

Ex. (68) Find in succession the cubes of $x^3 - 5x^2$, $x^3 - 5x^2 + 4x$, $x^3 - 5x^2 + 4x - 3$.

Ex. (69) Find by the above process the cube of $x^3 + 3x^2 + 4x + 5$.

Ex. (70) Check the final result obtained in Ex. 67 by cubing $- 3 + 4x - 5x^2 + x^3$.

Ex. (71) Find by the process used in (ii) the cube of 83793.

Ex. (72) Find also the cubes of 83782 and 84792.

32. The method may be used to extend a table of cubes. Thus, being given that $(987)^2 = 974169$, $(987)^3 = 961504803$, we can find the cube of 987·5.

987	974169	961504803
2961·5	2922507	962966796·875
	2923987·75	

Ex. (73) Given $(986)^2 = 972196$,
$(986)^3 = 958585256$;
find in the above manner the cube of 986·5.

Ex. (74) Find the cubes of 987·53 and of 987·534.

33. It is easy to extend the method to the evaluation of such algebraical expressions as $x^3 + 3x^2 + 8x + 4$, $2x^3 + 5x^2 + 11x + 2$, $3x^3 + 87x^2 + 514x + 6173$, etc., when x is a number of several digits. As in the evaluation of quadratic expressions the student will be left to infer the method by a careful examination of worked examples.

(i) Evaluation of $x^3 + 3x^2 + 8x + 4$ when $x = 567$.

1	3	8	4
	500	251500	125754000
	503	251508	125754004
	500	501500	50807280
	1003	753008	176561284
	500	93780	6691986
	1503	846788	183253270
	60	97380	
	1563	944168	
	60	11830	
	1623	955998	
	60		
	1683		
	7		
	1690		

Here from 125754004 (the value of $x^3 + 3x^2 + 8x + 4$ when $x = 500$) we form in succession the values 176561284, 183253270 of the same expression when $x = 560$ and when $x = 567$ respectively.

The same work, contracted, would stand thus:—

1	3	8	4
	503	251508	125754004
	1003	753008	176561284
	1503	846788	183253270
	1563	944168	
	1623	955998	
	1683		
	1690		

Ex. (75) Evaluate $x^3 + 3x^2 + 8x + 4$ when $x = 569$.
Ex. (76) Evaluate $x^3 + 4x^2 + 8x + 3$ when $x = 567$.
(ii) Evaluation of $2x^3 + 5x^2 + 11x + 2$ when $x = 567$.

2	5	11	2
	1005	502511	251255502
	2005	1505011	352806162
	3005	1692511	366182210
	3125	1887211	
	3245	1910864	
	3365		
	3379		

Ex. (77) Evaluate $3x^3 + 87x^2 + 514x + 6173$ by the above method when $x = 567$.

(iii) Evaluation of $x^3 + 3x^2 + 8x + 4$ when $x = 5{\cdot}673$.

1	3	8	4
	8	48	244
	13	113	318·496
	18·6	124·16	328·090963
	19·2	135·68	328·506543217
	19·87	137·0709	
	19·94	138·4667	
	20·013	138·526739	

Ex. (78) Evaluate $x^3 + 4x^2 + 7x + 5$ (i) when $x = 5{\cdot}673$, (ii) when $x = 3{\cdot}567$.

Ex. (79) Evaluate $x^3 + 3x^2 + 8x + 4$ when $x = 2{\cdot}468$.

Ex. (80) Evaluate $2x^3 + 5x^2 + 11x + 2$ (i) when $x = 56{\cdot}7$, (ii) when $x = 5{\cdot}67$, (iii) when $x = {\cdot}567$.

Ex. (81) Evaluate $5x^3 + 6x^2$ when $x = 3{\cdot}7951$.

Evolution.

34. We are now in a position to perform the inverse process to Involution. The ordinary method of extracting square root is well known, and nothing further need be said except that the multiplication and subtraction should be done in one operation as in the example worked on p. 5.

It should be noticed, however, that a *quadratic equation* may be solved by an extension of the square root method, just as the expression $2x^2 + 9x$ can be evaluated (see p. 28) when x is given by an extension of the method described for squaring.

Thus, being given that,

$$x^2 + 3x = 219490,$$

the positive value of x is found as follows :—

$$
\begin{array}{rrr}
1 & 3 & 219490 \\
 & \underline{400} & \underline{161200} \\
 & 403 & 58290 \\
 & \underline{400} & \underline{51780} \\
 & 803 & 6510 \\
 & \underline{60} & \underline{6510} \\
 & 863 & \cdots\cdot \\
 & \underline{60} & \\
 & 923 & \\
 & \underline{7} & \\
 & 930 &
\end{array}
$$

Here the work is of exactly the same nature as if we were evaluating $x^2 + 3x$ when $x = 467$.

400 is taken to begin with, since a rough trial shows that a value of x lies between 400 and 500; the 60 is taken next as the rough quotient of the remainder 58290 by 803 as a trial divisor; the 7 next as the rough quotient of the remainder 6510 by 923 as a trial divisor. 58290, 6510 and 0 being the respective remainders when the values of $x^2 + 3x$ for $x = 400$, $x = 460$, $x = 467$ are subtracted from 2192490. Hence the positive value of $x = 467$.

The same work contracted would stand thus:—

1	3	219490
	403	58290
	803	6510
	863	
	923	
	930	

Ex. (82) Solve, by the above method, the quadratic equations:—

(i) $x^2 + 4x = 2496$.

(ii) $x^2 - 3x = 2348554$.

(iii) $x^2 + 5x = 10676550$.

(iv) $x^2 - 5x = 22805394$.

The solutions of $2x^2 + 5x = 1174275$, $5x^2 + 97x = 29419632$, $4617x^2 + 29318x = 2821796{\cdot}0625$ are worked below, in the contracted form, as additional examples.

(i) $2x^2 + 5x = 1174275$.

2	5	1174275
	1405	190775
	2805	15275
	2925	. . . .
	3045	
	3055	

Here 190775, 15275, 0 are the respective remainders when the values $2x^2 + 5x$ for $x = 700$, $x = 760$, $x = 765$ are subtracted from 1174275. Hence $x = 765$.

(ii) $5x^2 + 97x = 29419632$.

5	97	29419632
	10097	9225632
	20097	386832
	22097	145362
	24097	. . . .
	24147	
	24197	
	24227	

Here 9225632, 386832, 145362, 0 are the respective remainders when the values of $5x^2 + 97x$ for $x = 2000$, $x = 2400$, $x = 2410$, $x = 2416$ are subtracted from 29419632. Hence $x = 2416$.

(iii) $4617x^2 + 29318x = 2821796{\cdot}0625$.

4617	29318	2821796·0625
	121658	388636·0625
	213998	170021·0625
	218615	11496·3325
	223232	. . . .
	226463·9	
	229695·8	
	229926·65	

$$x = 21{\cdot}75.$$

Ex. (83) Solve the equations :—

(i) $2x^2 + 5x = 129283$.

(ii) $2x^2 + 7x = 11014465$.

(iii) $3x^2 + 87x = 18383670$.

(iv) $5x^2 + 97x = 64142404$.

(v) $4617x^2 + 29318x = 3116707$.

Cube Root.

35. Several more or less cumbrous methods are described in text-books for the extraction of the cube root of a given number, all depending on the identity :—

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The student is recommended to discard at once any other method he may have learned in favour of Horner's.

Like most of the others it depends on the following considerations :—

Having found a part a of the cube root $a + h$ of a given number N it is easy to see from the above identity :—

(i) that the remaining part h is $\prec \dfrac{N - a^3}{3a^2}$;

(ii) that h must satisfy the equation :—

$$\begin{aligned} N - a^3 &= 3a^2h + 3ah^2 + h^3 \\ &= (3a^2 + 3ah + h^2)h. \end{aligned}$$

Selecting some number $b \prec \dfrac{N - a^3}{3a^2}$ we calculate the value of $3a^2b + 3ab^2 + b^3$.

If $3a^2b + 3ab^2 + b^3 \succ N - a^3$, we make a fresh trial, selecting some number $\prec b$.

If $3a^2b + 3ab^2 + b^3 = N - a^3$, then b is the required value of h.

If $3a^2b + 3ab^2 + b^3 \prec N - a^3$, we know that b is a part of h, and proceed to find the remaining part c by a repetition of the above trial process.

(i) $c \prec \dfrac{N - (a + b)^3}{3(a + b)^2}$;

(ii) $N - (a + b)^3 = 3(a + b)^2c + 3(a + b)c^2 + c^3$, and so on.

Now the system of columns constructed on p. 31 for finding the cube of a given number enables us from given values of a and b to calculate quickly the values :—

(i) of $3a^2b + 3ab^2 + b^3$; (ii) of $3(a + b)^2$.

Hence a slight modification of it will enable us to effect the extraction of the root $a + b + c$ of a given number N instead of the formation of the cube N of a given number $a + b + c$. The scheme of work will stand thus :—

$$
\begin{array}{lll}
a & a & N \\
\underline{a} & \underline{2a^2} & \underline{a^3} \\
2a & 3a^2 & N - a^3 \\
\underline{a} & \underline{\quad + 3ab + b^2} & \underline{\quad 3a^2b + 3ab^2 + b^3} \\
3a + b & 3a^2 + 3ab + b^2 & N - (a + b)^2 \\
\underline{\quad b} & \underline{\quad 3ab + 2b^2} & \underline{\quad 3(a + b)^2c + 3(a + b)c^2 + c^3} \\
3a + 2b & 3(a + b)^2 & N - (a + b + c)^3 \\
\underline{\quad b} & \underline{\quad 3(a + b)c + c^2} & \\
3(a + b) + c & 3(a + b)^2 + 3(a + b)c + c^2 & \text{which is by supposition zero.}
\end{array}
$$

We now give the work in full, except for the omission of unnecessary cyphers, for the extraction of the cube root of 91125.

4	16	91125 (45
4	32	64
8	4800	27125
4	625	27125
125	5425	. . . .

Here we took $a = 40$ since 91 lies between 4^3 and 5^3 so that the root was known to lie between 40 and 50, and on taking $h = 5$, the rough quotient of 27125 by 4800, we found our selection justified.

Here $3a = 120$ $\qquad 3a^2 = 4800$

$3a + h = 125 \qquad (3a + h)h = 625$

$$3a^2 + 3ah + h^2 = 5425$$
$$(3a^2 + 3ah + h^2)h = 27125.$$

As a further example we extract the cube root of 92345408, which will contain three digits.

4	16	92,345,408 (452
4	32	64
8	4800	28345
4	625	27125
125	5425	1220408
5	650	1220408
130	607500	. . . . .
5	2704	
1352	610204	

Here in the first two stages which are devoted to getting an approximation in defect to the cube root 92345000, as before we take $a = 400$, $b = 50$.

In the next we take $a + b = 450$, $c = 2$.

The work should be contracted thus in actual practice by performing the addition and subtraction simultaneously with the multiplication.

4	16	91,125 (45
8	4800	27125
125	5425	. . . .

4	16	92,345,408 (452
8	4800	28345
125	5425	1220408
130	607500	. . . . .
1352	610204	

36. Just as a *quadratic equation* may be solved by an extension of the *square root method*, a *cubic equation* may be solved by an extension of the *cube root method*: the process is merely the *inverse* of that of evaluating a cubic expression such as $2x^3 + 5x^2 + 11x + 2$.

Thus, being given that

$$2x^3 + 5x^2 + 11x = 360408972,$$

the positive value of x is found as follows :—

2	5	11	360408972 (564
	1005	502511	109153472
	2005	1505011	7602812
	3005	1692511	
	3125	1887211	
	3245	1900703	
	3365		
	3373		

Here the work is of exactly the same kind as if we were evaluating $2x^3 + 5x^2 + 11x$ when $x = 564$.

500 is taken to begin with, since a rough trial shows that a value of x lies between 500 and 600; the 60 is taken next as the rough quotient of the remainder 109153472 by 1505011 as a trial divisor; the 4 next as the rough quotient when the

remainder 7602812 is divided by 1887211 as a trial divisor; 109153472, 7602812, and 0 being the respective remainders when the values of $2x^3 + 5x^2 + 11x$ for $x = 500$, $x = 560$, $x = 564$ are subtracted from 360408972. Compare pp. 33, 34.

Ex. (84) Solve the equations:—

(i) $x^3 + 3x^2 + 8x = 1907238$.
(ii) $x^3 + 3x^2 + 8x = 14642{\cdot}784$.
(iii) $x^3 + 4x^2 + 8x = 116{\cdot}273625$.
(iv) $2x^3 + 5x^2 + 11x = 6{\cdot}245317632$.
(v) $3x^3 + 87x^2 + 514x = 29{\cdot}424042282789$.

II. ABBREVIATION AND APPROXIMATION.

37. In physical calculations we have generally to be satisfied with approximate results.

(1) Because some of the numbers we use have been obtained by observation and experiment, in which perfect accuracy is unattainable.

(2) Because some of the numbers we use, such as $\sqrt{5}$, and the numerical constants π, ϵ, etc., are incommensurable, *i.e.*, cannot be expressed exactly either as vulgar or decimal fractions.

There is indeed an important distinction between these two cases.

(1) An experimenter will soon find that there is a degree of accuracy beyond which the present state of the arts and sciences do not allow him to determine the values of the constants he is investigating. The following extract from Glazebrook's *Practical Physics* affords a good illustration: "If we set down the mechanical equivalent of heat at $4{\cdot}2 \times 10^7$ ergs, it is not because the figures in the decimal places beyond the 2 are all zero, but because we do not know what their values really are; or, it may be, for the purpose for which we are using the value, it is immaterial

what they are. It is known, as a matter of fact, that a more accurate value is $4{\cdot}214 \times 10^7$, but at present *no one has been able to determine what figure should be put in the decimal place after the 4.*

"The determination of an additional figure representing the magnitude of a physical quantity generally involves a very great increase in the care and labour which must be bestowed on the determination."

The student would do well to read the whole chapter (headed *Physical Arithmetic*) from which the above extract is taken.

(2) On the other hand, in the determination of approximate values of $\sqrt{5}$, π, ϵ, etc., there is no limit to the degree of accuracy attainable, which is merely a question of the time, arithmetical skill, and industry of the computer. For instance, the famous Abraham Sharp, about 200 years ago, calculated the logarithms of all prime numbers under 1100 to 61 places; the same "ingenious gentleman and indefatigable mathematician" found $\sqrt{12}$ to 72 places as a basis for the calculation of π to about the same number; and the evaluation of this last constant has been carried to at least 135 places by subsequent computers. No such accuracy is desirable when these constants make their appearance in physical formulæ; the computer must remember that *the result of his calculations cannot, except by accident, have a higher degree of accuracy than the experimental data on which they are founded,* and that he may save himself the trouble of carrying his preliminary numerical approximations to a much higher degree of accuracy than that of the physical ones with which he is going to use them.

38. We must in all cases be able to know to what extent we may rely on our final result. Some general remarks on the limits of error will be given in a future chapter. At present each case will be examined separately.

E.g., to find an approximate value of—

$$\tfrac{94}{95} + \tfrac{95}{96} + \tfrac{96}{97} + \tfrac{97}{98} + \tfrac{98}{99}.$$

$$\begin{aligned}
\tfrac{94}{95} &= \cdot 989473684\ \ldots \\
\tfrac{95}{96} &= \cdot 989583333\ \ldots \\
\tfrac{96}{97} &= \cdot 989690721\ \ldots \\
\tfrac{97}{98} &= \cdot 989795918\ \ldots \\
\tfrac{98}{99} &= \underline{\cdot 989898989}\ \ldots \\
& \ 4\cdot 948442645
\end{aligned}$$

It is, of course, clear that the last digit, 5, in this result is incorrect; and that the last but one, 4, may be incorrect from a wrong figure having been carried. But if we increased each digit in the last column by unity the sum of that column would, as 5 numbers are added, only be increased by 5, and the whole sum would consequently be 4·948442650; but to obtain this we took each decimal in excess of the corresponding vulgar fraction; hence the sum lies between 4·94844264
and 4·94844265
hence we know its value correctly to 7 places of decimals, viz., 4·9484426.

39. The student's attention is drawn to the meaning of the phrase: "*true to 2, 3, 4 . . . places of decimals*". Taking the last result as an example, if we were told to give an approximate value of $\tfrac{94}{95} + \tfrac{95}{96} + \tfrac{96}{97} + \tfrac{97}{98} + \tfrac{98}{99}$ correct to 2 places of decimals, we should give 4·95. For the true value lies between 4·94 and 4·95, so that either of these two values is nearer to it than any other we could write with only two decimal places; and whereas 4·94 falls short of the true value by more than ·008, and 4·95 exceeds the true value by less than ·002, therefore 4·95 is nearer the exact truth than 4·94. Similarly 4·948, 4·9484, 4·94844 and 4·948443 are successive approximations true to 3, 4, 5, and 6 places of decimals respectively. The only case in which

the student will find any difficulty as to whether he is to increase the last digit retained by unity, or to write it down unchanged, is when the first rejected digit following it is a 5.

Now if there are any more digits following the 5 it is plain that we obtain the more correct result by the first course; thus 6·377 is nearer to the true value of 6·37651 than 6·376 is.

But if the 5 is the final digit he will apparently be equally near the true value whichever course he takes; thus 6·376 and 6·377 are equally correct approximations to the value of 6·3765.

When such cases occur it will generally be a matter of indifference which we do, but it may be noticed that in most cases where a final 5 occurs *it is itself only an approximation*, and that a little examination of the previous work will show whether it was an approximation in excess or in defect of the true value; thus, to return to our calculated approximation to the value of $\frac{94}{95} + \frac{95}{96} + \frac{96}{97} + \frac{97}{98} + \frac{98}{99}$, if a student were, from the value 4·95, correct to 2 places, to write down a value correct to 1 place, he might regard 5 and 4·9 as equally correct approximations. But, taking the value 4·948 correct to 3 places, we see that 4·9 is rather the nearer of the two.

Absolute Error and Relative Error.

40. The "absolute error" in an approximation is the actual difference between the approximate value and the true value.

If we knew the amount of absolute error we should of course be able to obtain the true value, by addition or subtraction, from the approximation.

All that we know, however, in general, is that the absolute error in defect or excess is less than some definite

small number, and therefore that the true value lies between two definite values which can be found, by addition and subtraction, from the approximation. Thus the absolute error in the approximation 17·4807682 to the value of $\sqrt{7} + \sqrt{17} + \sqrt{23} + \sqrt{35}$ is shown to be less than ·0000002; hence the true value lay between 17·4807680 and 17·4807684 (see p. 49).

Even if we happen to know in which direction the error lies we shall frequently have to content ourselves with the knowledge that it lies between two definite limits.

For instance, the value of π is known to be greater than 3·14159; whereas the approximate value $\frac{22}{7}$, frequently used for the sake of convenience, = 3·142857 . . . Hence we know that the absolute error in this approximation

$$\begin{aligned} &\not< 3{\cdot}1428 - 3{\cdot}1416 \\ \text{but } &\not> 3{\cdot}1429 - 3{\cdot}1415 \end{aligned}$$

and therefore lies between ·0012 and ·0014.

41. We are generally, however, less concerned with the "absolute" than with the "relative" error in an approximation. This is especially the case when we have to find approximate products or quotients (see pp. 63, 69).

The "relative error" in an approximation is the ratio of the "absolute error" to the true value.

From what has just been said about the absolute error it can be seen that the relative error can in general only be shown to lie between certain limits.

In the first example on p. 49,

$$\begin{aligned} \text{since absolute error} &\not> {\cdot}0000002, \\ \text{and true value} &\not< 17{\cdot}4807678, \\ \therefore \text{ relative error} &\not> \frac{{\cdot}0000002}{17{\cdot}4807678}. \end{aligned}$$

This is an "upper limit" to the relative error. If we had known that the absolute error was *greater* than some

definite quantity we could in a similar way have assigned a "lower limit".

Thus, in the next example chosen, when we take $\frac{22}{7}$ for π,

$$\text{relative error} \prec \frac{\cdot 0014}{3\cdot 14159} \text{ but} \succ \frac{\cdot 0012}{3\cdot 1416}.$$

In practice it is frequently unnecessary to assign the limits of relative error so narrowly as in the above example. For many purposes it would be sufficient to give:—

$$\text{relative error} \prec \frac{\cdot 0014}{3} \text{ but} \succ \frac{\cdot 0012}{4},$$

$$\text{or even} \prec \cdot 0005 \text{ but} \succ \cdot 0003.$$

As another example, suppose that the length of a metre is known, *true to four significant figures*, to be 39·37 inches.

Here absolute error $\prec$ ·005 inches.

$$\therefore \text{ relative error} \prec \frac{\cdot 005}{30},$$

$$\text{and } \therefore \prec \cdot 0002.$$

It is sometimes convenient to find, besides the "upper limit," some simple fraction that the relative error may possibly reach. Thus, in the above example, since the absolute error might be greater than ·004, the relative error might be greater than $\frac{\cdot 004}{40}$, *i.e.*, than ·0001.

If, however, instead of simply knowing that 39·37 was true to four significant figures, we had known that 39·37079 in. was the value true to 7 significant figures, we could have assigned closer limits to the error made in using 39·37 in.

Since absolute error in defect $\prec$ ·0008,

$$\therefore \text{ relative error in defect} \prec \frac{\cdot 0008}{30}, \text{ and } \therefore \prec \cdot 00003.$$

Also, since absolute error in defect $\succ$ ·0007,

$$\therefore \text{ relative error in defect} \succ \frac{\cdot 0007}{40}, \text{ and } \therefore \succ \cdot 00001.$$

Sometimes a rate of error is given as a percentage. It

is plain from the definition of "relative error" that percentage error = 100 times the relative error.

42. It should be carefully noticed that while absolute error is increased by multiplying by an exact factor, the relative error is left unaltered. Thus while the absolute error in taking 100 metres as 3937 in. is 100 times that in taking 1 metre as 39·37 in., the relative error is just the same as before.

Ex. (85) Find between what limits, to one significant figure, the relative error lies in taking $\frac{1307}{247}$ for $\sqrt{28}$.

Ex. (86) Find to two significant figures the relative error in taking (i) $\frac{355}{113}$, (ii) $\frac{103993}{33102}$, for 3·1415926535 . . .

43. Relative Error has been defined as the fraction—

$$\frac{\text{Absolute Error}}{\text{True Value}}.$$

It will be easily seen that this fraction differs very slightly from—

$$\frac{\text{Absolute Error}}{\text{Approximate Value}}$$

which is often used for it.

Now, it has been pointed out that in general it is impossible to find the exact value of the first fraction, and that we have to content ourselves with assigning upper and lower limits to it.

44. In the great majority of cases that will come under the student's notice, the simple quantities which in approximate calculations he would naturally select for limits to the first fraction, are also limits to the second. But it is just possible that this may not be the case. For instance, suppose the first fraction = ·00003998 . . .

and the second = ·00004001 . . .

it is plain that ·00004 would be an upper limit to the first but not to the second; it is, however, also plain that a

simple value ·000041 can be written down which is an upper limit to each of them.

In what follows, when we speak of upper and lower limits of relative error, we shall suppose them so taken that they are also upper and lower limits of the fraction—

$$\frac{\text{Absolute Error}}{\text{Approximate Value}}.$$

Approximation in Addition.

45. Suppose it is required to find the sum of $\sqrt{7}$, $\sqrt{17}$, $\sqrt{23}$, $\sqrt{35}$ by means of a table of square roots.

$$\left.\begin{aligned}\sqrt{7} &= 2{\cdot}6457513\\ \sqrt{17} &= 4{\cdot}1231056\\ \sqrt{23} &= 4{\cdot}7958315\\ \sqrt{35} &= \underline{5{\cdot}9160798}\end{aligned}\right\}\text{true to 7 places of decimals.}$$
$$17{\cdot}4807682$$

Here each of the four approximate values, being taken from a table "true to 7 places," differs from the corresponding true one by a fraction less than ·000,000,05, either in excess or in defect. Hence the total error, either in excess or in defect, is less than ·0000002.

Hence the required sum must lie between 17·4807684 and 17·4807680; and its value, "true to 6 places," is 17·480768.

It will be an instructive exercise to take these four values true to 6, 5, 4, 3, 2 places of decimals in succession, and examine how far we can depend on the value of the sum.

True to 6 places.	5 places.	4 places.	3 places.	2 places.
2·645751	2·64575	2·6458	2·646	2·65
4·123106	4·12311	4·1231	4·123	4·12
4·795832	4·79583	4·7958	4·796	4·80
5·916080	5·91608	5·9161	5·916	5·92
17·480769	17·48077	17·4808	17·481	17·49

Hence, by the reasoning used above, the true value is known in the several cases to lie between—

	17·480770	17·48079	17·4810	17·483	17·51
and	17·480766	17·48075	17·4806	17·479	17·47.

Hence to 5 places, 4 places, 3 places, 2 places, 1 place, its value is—

17·48077 17·4808 17·481 17·48 17·5.

We notice here that the addition enabled us in each case to obtain a result true to a number of places one less than that of the approximation added.

This had to be determined by the examination of each case, and was not obvious beforehand. In fact, if we abbreviate still further, and retain only one place of decimals, we can only infer that the result is between 17·6 and 17·2, and are left in doubt as to whether 17 or 18 is the nearer integral value.

2·6
4·1
4·8
5·9
17·4

46. A special case of addition that often occurs in physical research is that of a number of approximations to the same unknown true value in order to find their "mean" or "average".

In this case the results will all agree to a certain number of digits, and the average can be found without doing the whole of the addition.

We select an example from Joule's *Scientific Papers* in which the specific gravity of tin is required.

Six separate sets of investigation having given as results 7·291, 7·290, 7·291, 7·295, 7·285, 7·248, we want to find the result of dividing the sum of these numbers by 6.

Here it is plain that 7·2 will be the first two digits in the result, and we need only do the addition and division for the third and fourth digits thus :—

7·291
7·290
7·291
7·295
7·285
7·248

6) 500

7·283

Here the sum of the last two columns being 500 the average was 83, to which the 7·2 was prefixed.

Ex. (87) (i) Find the average of $\frac{94}{95}$, $\frac{95}{96}$, $\frac{96}{97}$, $\frac{97}{98}$, $\frac{98}{99}$, to 8 significant figures; (ii) the absolute error in taking $\frac{96}{97}$ instead of this average; (iii) what is the relative error in the same case.

47. The following theorems, proved in treatises on Algebra, are useful in estimating the relative error in the sum of a number of approximate terms. The demonstrations are not difficult, all the letters being supposed to denote positive quantities.

(1) If the fractions $\frac{a}{A}$, $\frac{b}{B}$, $\frac{c}{C}$, $\frac{d}{D}$ are all equal, the fraction

$$\frac{a+b+c+d}{A+B+C+D}$$

is equal to each of them.

(2) If the fractions $\frac{a}{A}$, $\frac{b}{B}$, $\frac{c}{C}$, $\frac{d}{D}$ are not all equal, the fraction

$$\frac{a+b+c+d}{A+B+C+D}$$

is intermediate in value between the greatest and least of them.

Now, suppose that a, b, c, d are the errors made in approximating to the true values, A, B, C, D, and suppose

that the approximations are either all in excess or all in defect, then the absolute error made in the sum will be $a + b + c + d$, and consequently the relative will be:—

$$\frac{a + b + c + d}{A + B + C + D}.$$

If all the relative errors $\frac{a}{A}, \frac{b}{B}, \frac{c}{C}, \frac{d}{D}$ are equal, then this is equal to each of them. If they are not, it is less than the greatest of them.

Now, if approximations were some in excess and some in defect, it is plain that the absolute error would be less than $a+b+c+d$, and hence that the relative would be less than—

$$\frac{a + b + c + d}{A + B + C + D}.$$

Hence in all cases the largest relative error of the terms to be added is an upper limit to the relative error of the result.

Approximation in Subtraction.

48. Suppose it is required to find the value of $\frac{1}{847} - \frac{1}{953}$ by means of a table of reciprocals.

$$\begin{aligned} \tfrac{1}{847} &= \cdot 0011806 \\ \tfrac{1}{953} &= \underline{\cdot 0010493} \\ & \cdot 0001313 \end{aligned}$$

Here, as each result is "true to 7 places," we know that it cannot differ by so much as ·00000005 from the true; but each may be in excess or defect of the true value: we can therefore only infer that the result of the subtraction does not exceed or fall short of the true difference by so much as ·0000001.

Hence the true result lies between ·0001314 and ·0001312, and its value true to 6 places of decimals is ·000131.

49. It should be carefully noticed that the relative error in the difference of two approximations may be very much greater than the relative error of either of the approximations considered separately. This fact is especially to be borne in mind in physical investigations. See Glazebrook's *Practical Physics*, pp. 48, 49.

Consider, for instance, the relative error in the approximation to $\frac{17}{12} - \sqrt{2}$ found by taking the difference of the approximations of these true to 5 significant figures.

$$\left.\begin{aligned}\tfrac{17}{12} &= 1{\cdot}4167 \\ \sqrt{2} &= \underline{1{\cdot}4142}\end{aligned}\right\}\text{to 4 places of decimals.}$$
$$\cdot 0025$$

Now by taking each out to a greater number of decimal places it can be shown that—

Relative error in first approximation $\prec \cdot 00002$
Relative error in second approximation $\prec \cdot 00004$
Absolute error in excess in result $\succ \cdot 00004$

Hence relative error in result $\succ \dfrac{\cdot 00004}{\cdot 0025}$

and $\therefore \succ \cdot 016$

Ex. (88) Find the relative error (i) in $\sqrt{3} + \sqrt{2}$, (ii) in $\sqrt{3} - \sqrt{2}$ when 1·41421 and 1·7321 are taken for $\sqrt{2}$ and $\sqrt{3}$ respectively.

Approximation in Multiplication.

50. Three cases present themselves for consideration. Of the two factors to be multiplied together:—

(i) Both may be known exactly.

(ii) One may be known exactly and the other only approximately.

(iii) Both may be only known approximately.

(i) Suppose it be required to find the product of the two exact factors, 258·73 and 24·568, true to 5 significant figures. As there will clearly be 4 integral digits in the product, this is equivalent to finding the product true to 1 place of decimals. In order that the student may the better understand the contraction, the work will first be done in full, in the manner previously explained.

```
  258·73
   24·568
  ------
5174·6
1034·92
  129·365
   15·5238
    2·06984
-----------
6356·47864
```

The reasoning applied to the examples worked out under the head of "Approximate Addition" will enable the student to see that, in most cases that are likely to occur, it would be sufficient if he knew each of the partial products which are to be added up to obtain the final result "true to 3 places of decimals". We will write them down "true to $\underline{3}$ places" while performing the multiplication.

```
 258·ˎ7ˎ3
  24·568
 --------
5174·6
1034·92
 129·365
  15·524
   2·070
 --------
6356·5
```

We proceed as before, carefully pointing off the first partial product, until we have completed the multiplication by the 5.

The multiplication by 6 would give us a figure in the fourth place of decimals. Hence mark off the final 3 thus, ˎ3; and *instead of setting down the result of multiplying it by the 6, carry the nearest* $\underline{10}$ *of that result to the next column, thus :—*

6 times 3, 18, carry $\underline{2}$ (because 1·8 = 2 app.).

6 times 7, 42; and 2, 4′$\underline{4}$, etc.

Having finished multiplying by 6 we begin multiplying by 8, having previously marked off the digit 7 next to the 3, thus ‸7.

8 times 7, 56, carry $\underline{6}$ (because 5·6 = 6 app.).

8 times 8, 64; and 6, 7'$\underline{0}$, etc.

On performing the addition we do not set down the results for the second and third places of decimals, but notice that the "true" figure to carry to the first is 2.

On account of the importance of the method another example is given: *To find the product of 5487·324 by 128·57246 true to 2 places of decimals.*

548‸7·‸3‸2‸4
128·57246
548732·4
109746·48
43898·592
2743·6620
384·1127
10·9746
2·1949
·3292
705518·75

Here 4, 2, 3, 7 are cut off successively from the multiplicand, when we begin to multiply by 7, 2, 4, 6 respectively. The sum of the digits in the column on the extreme right is 24; as four of these digits are only approximations we should be in excess of the true value by taking it as 26, and in defect by taking it 22.

According as we carry 2 or 3 to the next column the sum becomes 25 or 26: the first of these values being in defect, the second in excess.

Hence, if we carry 3 to the next column (that of the second place of decimals) we shall obtain a result true to that number of decimals.

Ex. (89) Find the following products true (*a*) to 5, (*b*) to 7 significant figures.

(*a*) (i) 2·4142136 × 3·7320508.

(ii) $(3·14159)^2$.

(iii) 5·645751 × 2·354249.

(iv) 5·795832 × 2·204168.

(v) 3·316625 × 2·236068.

(*b*) (i) 5487·3244 × 128·572464.

(ii) 5487·3235 × 128·572455.

(iii) 5487·3238 × 128·572463.

Ex. (90) Find the following products true to 5 places of decimals.

(i) $(3\cdot100628)^2$; (ii) $(9\cdot7409089)^2$; (iii) $\cdot030602 \times \cdot310063$; (iv) $\cdot2588190 \times 9\cdot659258$; (v) $1\cdot736482 \times 9\cdot848078$.

51. (ii) Suppose one of the factors is known exactly and the other only approximately.

We will first take the simplest possible case, viz., when the exact factor is represented by a single digit.

Express 6 metres in inches, having given that one metre = 39·37 in. approximately.

$$\begin{array}{r} 39\cdot37 \\ 6 \\ \hline 236\cdot22 \end{array}$$

As we are only supposed to know that the given approximate value of a metre is not more than ·005 of an inch in excess or defect, and as $\cdot005 \times 6 = \cdot03$, we can only infer that 6 metres is not less than 236·19 in. and not greater than 236·25 in.

Similarly if we are given that 1 gallon = 4·5435 litres, we can infer that—

8 gallons is not less than 36·3476 litres,
and not greater than 36·3484.
Hence 8 gallons = 36·348 litres app.

The student should notice that the product cannot be found true to a greater number of figures than the approximate value given.

Ex. (91) Make a table of multiples of a gallon up to 9 times true to 3 decimal places in litres.

Ex. (92) Given 1 inch = 2·53995 centimetres.
1 litre = 1·76077 pints.
1 gramme = 15·43235 grains.
1 kilometre = ·621382 miles.
1 metre = 39·37079 inches.

Make tables of multiples up to 9 times, of an inch, a litre, a gramme, a kilometre, a metre, in centimetres, pints, grains, miles, inches, respectively.

52. Proceeding to cases where the exact factor has two digits, suppose it is required to find the value of 6·4 metres in inches.

39·37	6·4	Limit to amount of error in excess
6·4	·005	or defect.
236·22	·0320	
15·748		
251·968		
·032		
252·0	by adding.	
251·9	by subtracting.	

We see that 6·4 metres = 252 inches true *to the units figure*, the error in excess being less than ·1.

Ex. (93) Find as to as many significant figures as possible 123 gallons in litres, 234 inches in centimetres, 345 litres in pints, 456 grammes in grains, 567 kilometres in miles, 678 metres in inches, from the approximations given in Ex. 92.

53. When we happen to know on which side of the true value the given approximation lies, the true value of the product will be known within narrower limits than have been assigned above. Thus in the above example a student, who has previously seen the value of a metre given as 39·37079 inches, although unable to remember all the digits in the decimal fraction, will perhaps remember the first four; he will therefore know (1) that 39·37 is in defect; (2) that the defect is less than ·005 of an inch (otherwise the four figure value would have been 39·38).

$$\text{Hence } 6{\cdot}4 \text{ metres lies between } \left.\begin{array}{r}252{\cdot}\\ \text{and } 251{\cdot}968\end{array}\right\}\text{inches,}$$

and the approximate value 252 obtained before is known to be true to *the first place of decimals*, the error in excess being less than ·04 of an inch.

54. A consideration of the "relative error," in the given approximation, will often enable us to obtain a limit to the absolute error in the product, sufficiently near for our purpose without performing the little calculations like that (6·4 × ·005) on the right-hand side of p. 57.

Thus the "relative error," when a metre is taken as 39·37 inches, is obviously less than $\frac{\cdot 005}{30 \cdot 000}$, *i.e.*, than $\frac{1}{6000}$.

Hence having found that—

$$39 \cdot 37 \times 6 \cdot 4 = 251 \cdot 968,$$

we see that the absolute error in the product must be less than $\frac{251 \cdot 968}{6000}$ *i.e.*, $\prec \cdot 0419$, etc.

$$\begin{array}{ll} \text{First approximation} & = 251 \cdot 968. \\ \text{Error} & \prec \quad \cdot 042. \end{array}$$

Hence if we are in doubt whether 39·37 is "in excess" or "in defect" we can only see that the true value lies between 252·010 and 251·926, and hence that, *true to units*, it is 252.

If we happen to know that 39·37 is "in defect," we should see that the true value lay between 252·010 and 251·968, and hence that, *true to the first place of decimals*, it would be 252·0.

As it is important that the student should master this method of considering the relative error before proceeding to the next section, another fully worked example is added.

Express 3228 watts in horse-power, given a watt = ·001341 H.-P.

$$\begin{array}{r} 3228 \\ \cdot 00134 \\ \hline 3 \cdot 228 \\ \cdot 9684 \\ \cdot 1291 \\ \hline 4 \cdot 326 \end{array}$$

Here the relative error $\prec \dfrac{\cdot 000005}{\cdot 001}$,

$$i.e., \prec \frac{1}{200}.$$

$\therefore$ absolute error $\prec \cdot 022$.

3228 watts lies between 4·35 and 4·30 H.-P.

Here we were not supposed to know anything about the given approximate value of the watt except that it was true to the three significant figures given.

If we happen to know that it lay between the given value and ·001341 we should see that—

$$\text{relative error} \prec \frac{1}{1000},$$

$$\therefore \text{ absolute error} \prec \cdot 00044.$$

Hence to two places of decimals—

$$3228 \text{ watts} = 4 \cdot 33 \text{ H.-P.}$$

55. (iii) Lastly let it be required to find the product of two factors, each of which is only known approximately (such as π and $\sqrt{3}$).*

The student is recommended to read again carefully the remarks, at the commencement of Section II., as to the distinction between *physical constants* (such as the *mechanical equivalent of heat* or the *length of a second's pendulum*), in which there is a limit to the degree of attainable accuracy, and *numerical constants* (such as π or ϵ), in which there is no such limit.

The values of most of the latter may be looked upon as known accurately, since they have been calculated and tabu-

* This must always be the case when the two factors express the measure of concrete quantities.

lated to a degree of accuracy far beyond that attainable in the laboratory or workshop, and all we have to do when the product of any two of them is required is to take each factor from the tables to such a number of places that the error does not affect the accuracy of the product to the number of places desired.

56. It will greatly help the student to form a sound opinion as to the number of places to which he should write down the factors, in order to secure the correctness of the product to a given number of places, if he works out a large number of sets of examples in which the same factors are carried out to different numbers of places.

Consider the calculation on p. 55, where the product of 5487·324 and 128·57246 has been found true to eight significant figures on the supposition that these factors were exact, and examine how the product is affected by the supposition that the factors are approximations merely true to the number of places given. All we can be sure of is that the multiplicand lies between 5487·3235 and 5487·3245, and the multiplier between 128·572455 and 128·572465, and a very slight examination will give the student good reason for feeling doubtful about the seventh and eighth figures in the product.

We will calculate the product to the same number of places with both the upper and lower limit of the two factors.

5487·3245	5487·3235
128·572465	128·572455
548732·45	548732·35
109746·490	109746·470
43898·5960	43898·5880
2743·6623	2743·6618
384·1127	384·1126
10·9746	10·9746
2·1949	2·1949
·3292	.2744
·0274	·0274
705518·84	705518·65

Hence while we are certain that the product true to six significant figures is 705519, we cannot be certain as to the seventh figure, though it must be either 7 or 8.

57. It may perhaps occur to the student—

(i) That, as we can only find the product of the two numbers in question true to units, we did an unnecessary amount of work in carrying out the partial products to four places of decimals.

(ii) That the multiplier, which happens to be given to one more place than the multiplicand, might have been cut down to the same number of places without affecting the accuracy of the units.

On examination he will be confirmed in both these conjectures. We append—

(i) The working of the same example with the partial products taken out to two places of decimals only.

(ii) The working of the same example with the multiplier true to six figures only.

(i)	(ii)
5487·324	5487·324
128·57246	128·5725
548732·4	548732·4
109746·48	109746·48
43898·59	43898·59
2743·66	2743·66
384·11	384·11
10·97	10·97
2·19	2·74
·32	
705518·72	705518·95

58. We now proceed to some general considerations on that amount of error possible in the product, which arises not from the process of contraction, but from the possible

errors in the factors multiplied together. It will in general be more important to find a *simple limit* in either sense to the possible error of the product, than to determine it with any great degree of precision.

Consideration of Absolute Error.

Let the true values of two factors to be multiplied be denoted by A and B, and the numbers which express them approximately by a and b. Further, suppose that each is known true to five significant figures, and that the *leading digit* * of each is in the units' place.

Then A lies between a + ·00005 and a − ·00005,

and B lies between b + ·00005 and b − ·00005.

Hence AB lies between—

$$ab + \cdot 00005\,(a + b) + \cdot 0000000025,$$
$$\text{and } ab - \cdot 00005\,(a + b) + \cdot 0000000025.$$

Now $a + b$ must in any case be greater than 2, and may be nearly 20; thus in using the product ab as an approximate value for AB, the error may show in the third decimal place.

It is obvious, therefore, that if we worked in full the product ab to eight decimal places, the last four, possibly the last five, would be of no use to us.

Suppose, for instance, that A stands for π, and B for $\sqrt{5}$; while a = 3·1416, and b = 2·2361.

Then ab = 7·02493176; while $a + b \prec 6$; and $(a + b)\,\cdot 00005 \prec \cdot 0003$.

But since $a + b \succ 5$ the error may amount to ·00005 × 5 (= ·00025) in either sense, so that the last three figures are of no use to us while the 93 is incorrect, though useful in conjunction with $\pm$ ·0003, in determining what figure should be put in the third decimal place. On examination we see that to five figures $\pi\sqrt{5} \prec 7{\cdot}0252$, but $\succ 7{\cdot}0246$.

* *i.e.*, the digit of highest denomination.

Hence to four figures $\pi\sqrt{5} = 7{\cdot}025$.

59. The method explained in the last paragraph may be easily extended. Thus (i) if 82·346 and 56·341 be values true to the third decimal place of two factors, the error in either sense made through taking the product of these for the product of the actual factors will be less than $140 \times {\cdot}0005$, but possibly greater than $130 \times {\cdot}0005$.

(ii) If 7439 and 2561 be the values true to units, the error in the product would be less than $10000 \times {\cdot}5$, but possibly $\succ 9000 \times {\cdot}5$.

(iii) If 346·72 and 54·837 be the values true to five figures, the error would be less than $90 \times {\cdot}005$ (or $900 \times {\cdot}0005$), but possibly greater than $80 \times {\cdot}005$.

Ex. (93*a*) Find the products in (i), (ii), and (iii), true to as many places as you can.

Consideration of Relative Error.

60. The following theorem is of great importance in considering the relative error of a product.

If there is a small error in each of two factors, the "relative error" in the product formed from them is approximately equal to the (algebraical) sum of the "relative errors" of the two factors.

It may happen that at the same time as we are given the approximate values of the factors, we are also given limits of the relative errors of those factors. It will in this case frequently be more advantageous to apply the theorem just quoted, than to use the method of the last section.

A short algebraical investigation may help the student to realise the truth of the theorem given, and to apply it in practical calculations.

Let the true values of two factors to be multiplied be denoted by A and B, and the numbers which express them approximately by a and b.

Further, let the approximations be in defect, and let their absolute errors be x and y.

Then $A = a + x$

and $B = b + y$.

Therefore $AB = ab + ay + bx + xy$.

Now x and y being small compared with a and b, it is clear that to a considerable number of significant figures (n suppose)—

$$\begin{aligned} AB &= ab + ay + bx \\ &= ab + ab\left(\frac{x}{a} + \frac{y}{b}\right). \end{aligned}$$

Let X and Y be upper limits of $\frac{x}{a}$ and $\frac{y}{b}$,

then, to n significant figures,

$$AB \prec ab + ab\,(X + Y).$$

On the other hand, if X and Y were lower limits, then to n significant figures—

$$AB \succ ab + ab\,(X + Y).$$

Similarly, if the approximations had been in excess, to n significant figures—

$$AB \gtrless ab - ab\,(X + Y),$$

according as X and Y are upper and lower limits of $\frac{x}{a}$ and $\frac{y}{b}$.

Suppose that to 4 significant figures, $A = 85{\cdot}46$ and $B = 4{\cdot}283$, and that these approximations are in defect.

Then $xy \prec {\cdot}0000025$.

Hence to four places of decimals (*i.e.*, to 7 significant figures)—

$$AB = ab + ab\left(\frac{x}{a} + \frac{y}{b}\right).$$

Suppose further that $\frac{x}{a}$ is known to lie between ·00003 and ·00005, and that $\frac{y}{b}$ lies between ·00006 and ·00007.

Then to any number of significant figures less than 7—

$$AB \prec ab + ab\,(\cdot 00012),$$
$$\text{but} \succ ab + ab\,(\cdot 00009).$$

Now ab clearly lies between 300 and 400, so that the error in taking ab for the product will show in the second decimal place.

Hence we shall not be able to get the product true to more than one place of decimals; we shall, however, to avoid errors from the process of contraction, take the product out to two places further.

$$\begin{array}{r}
85\cdot46 \\
4\cdot283 \\
\hline
341\cdot84 \\
17\cdot092 \\
6\cdot837 \\
\cdot256 \\
\hline
366\cdot025
\end{array}$$

Here the error arising from the contraction cannot amount to ·001 in either sense. (See p. 49.)

$$\begin{array}{rl}
\text{Hence AB} \succ & 366\cdot024 \\
+ & \cdot027\ (\cdot00009 \times 300) \\
\therefore\ \text{AB} \succ & 366\cdot05 \\
\text{Again AB} \prec & 366\cdot026 \\
+ & \cdot048\ (\cdot00012 \times 400) \\
\therefore\ \text{AB} \prec & 366\cdot08
\end{array}$$

while to 4 significant figures AB = 366·1.

In the following examples find the results true to as many significant figures as possible, with the use of as few figures as possible in the multiplication, and discuss the limits of error.

Ex. (94) Find the diagonal of a square in metres whose side is 10 yds., given that 1 yd. = ·91438 of a metre; $\sqrt{2} = 1{\cdot}4142$.

Ex. (95) Find the following products, the factors being supposed true to the number of significant figures given.

(i) 7·8562 × 5·3987.

(ii) 743·916 × ·0047865.

(iii) 4953·1 × 287·06.

(iv) 57342000 × 687490000.

(v) 2·875641 × 7·32148 × 8·96324.

Approximation in Division.

61. As in multiplication, three cases present themselves. Of the divisor and dividend :—

(i) Both may be known exactly.

(ii) One may be known exactly and the other only approximately.

(iii) Both may be known only approximately.

(i) Suppose we are required to divide 42856731275834 by 574238 correct to the tens' place.

```
                74632350
5,7,4,2,3,8 ) 42856731275834
                2660071
                 363119
                  18576
                   1349
                    201
                     29
```

Here, proceeding by the method explained on p. 17, the first digit in the quotient is 7 and the first remainder is 266007; bringing down the 1, the second digit is 4 and the remainder 363119.

We now notice that the remaining part of the quotient 275834 *is less than the divisor.*

At this stage it is convenient to stop bringing down any more figures, and obtain the remaining digits of the quotient by continually contracting the divisor. Thus, cutting off the 8, we see that 57423 is contained 6 times in 363119: also since 6 times 8 is 48, to which the nearest "ten" is 50, we carry 5 on to the 18 obtained by multiplying 3 by the 6.

Similarly—

5742 is contained 3 times in the next remainder 18576.
574 is contained twice in the next remainder 1349.
57 is contained 3 times in the next remainder 201.
5 is contained 5 times in the next remainder 29.

Here we have written down approximate values for the remainders, as we did on p. 61 for the partial products. Hence a certain amount of doubt hangs over the last digit 5 obtained in the quotient, which we proceed to examine. As in the corresponding case in multiplication the error cannot amount to so much as 5 in the next column to the one retained in any one case. Hence, as there are three approximations made before reaching the 29, the error does not accumulate either in excess or defect to 2 in the column of lowest order retained.

Taking the last remainder, either 31 or 27, it will be seen that 5 is the nearest digit for the last place, though whether it is in excess or in defect is uncertain. It will save this examination if we continued the ordinary process one place further before beginning with the abbreviations, thus:—

$$\begin{array}{rl}
 & \underline{74632350} \\
57_{\backslash}4_{\backslash}2_{\backslash}3_{\backslash}8\ ) & 42856731275834 \\
 & 2660071 \\
 & 3631192 \\
 & 185764 \\
 & 13493 \\
 & 2008 \\
 & 285
\end{array}$$

Here we see that 7463235 is in excess, since $57{\cdot}4 \times 5 \succ 285 +$ error possibly accumulated.

If the student finds any difficulty in doing the work in the way described above, he should go through it with the abbreviations recommended, but without using the "Italian" method of combining the subtraction and the multiplication.

For the sake of simplicity integral numbers were chosen for divisor and dividend, but the process can easily be applied to decimals.

Thus, to divide 62·473 by 419·6789, we first shift the point 4 places in each, so as to have an integral divisor; this renders the pointing off easier in the quotient; and then work as follows:—

$$\begin{array}{rl}
 & \underline{\cdot 1488590} \\
41_{\backslash}9_{\backslash}6_{\backslash}7_{\backslash}8_{\backslash}9 \;) & 624730{\cdot}0 \\
 & 20505110 \\
 & \;\;3717954 \\
 & \;\;\;360523 \\
 & \;\;\;\;24780 \\
 & \;\;\;\;\;3796 \\
 & \;\;\;\;\;\;\;19
\end{array}$$

Here the 1 and 4 were obtained without abbreviating, and the 8, 8, 5, 9, 0 were obtained by cutting off 9, 8, 7, 6, 9 in succession. Moreover, as an estimation of error shows that the remainder to the places retained lies between 21 and 17, we see that the quotient true to the eighth place is either ·14885905 or ·14885904.

Another worked example is added. Divide ·0167 by 423·74.

$$\begin{array}{rl}
 & \underline{\cdot 00003941} \\
42_{\backslash}3_{\backslash}7_{\backslash}4 \;) & 1{\cdot}67000 \\
 & \;39878 \\
 & \;\;1741 \\
 & \;\;\;\;46 \\
 & \;\;\;\;\;4
\end{array}$$

Examination of the possible error will show that the quotient, correct to the fifth significant figure, is ·000039411.

In most of such examples as are likely to occur, the student may, if his divisor is of n digits, obtain $n - 2$ correct digits by gradually abbreviating it from the beginning; the digit obtained by the next abbreviation will very likely also be correct, but it will be better for him to ascertain the amount of possible error, as in the examples just worked.

Ex. (96) Find to 5 significant figures the following quotients: (i) 5·645751 ÷ 2·354249; (ii) 1·4142135 ÷ 1·7320508; (iii) 13·8209163 ÷ ·0236753; (iv) 6·6332496 ÷ ·046904158.

Ex. (97) Find to 4 places of decimals:—

(i) 7·4161985 ÷ 3·31622777; (ii) 1 ÷ ·105263;
(iii) 9216 ÷ 8847·36; (iv) 80102·5 ÷ 716917·375;
(v) 851 ÷ 2·91719043.

62. (ii) and (iii) When either the divisor or the dividend is only known approximately, the degree of accuracy is best tested by estimating "relative error" as in the corresponding cases in multiplication, using the following theorem corresponding to the one given on p. 63.

The "relative error" of a quotient is found approximately by subtracting (algebraically) the "relative error" of the divisor from the "relative error" of the dividend.

Thus, *to find the diameter of a circle whose circumference is 100 feet; given that circumference = diameter × 3·1415927.*

$$\begin{array}{r} \underline{31}\cdot\underline{83} \\ 3\cdot1_\backslash4_\backslash1_\backslash5_\backslash9\ )\ 1000000 \\ 5752 \\ 2610 \\ 97 \end{array}$$

Here A. E. in 3·14159 $\prec$ ·000003,
∴ R. E. in 3·14159 $\prec$ ·000001,
∴ A. E. in 31·83 $\prec$ ·000032.

Hence to 4 significant figures diameter = 31·83.

Express "one litre per hectare" (irrigation) in "cubic inches per acre"; given 1 litre = 61·0793 cub. in.

$$1 \text{ hectare} = 2{\cdot}47255 \text{ ac.}$$

$$\begin{array}{r} \underline{24}{\cdot}\underline{703} \\ 2_{\backslash}4_{\backslash}7_{\backslash}2_{\backslash}5_{\backslash}{\cdot}5 \;)\; 610793 \\ 11628 \\ 1738 \\ 7 \end{array}$$

$$\begin{array}{ll} \text{Here R. E. in } 2{\cdot}47255 & \prec {\cdot}000003, \\ \text{and R. E. in } 61{\cdot}0793 & \prec {\cdot}000001, \\ \therefore \text{ R. E. in } 24{\cdot}703 & \prec {\cdot}000004, \\ \therefore \text{ A. E.} & \prec {\cdot}0001. \end{array}$$

∴ one litre per hectare = 24·70 cub. in. per acre true to 4 significant figures.

Express "1 cwt. per acre" (farm produce) in "kilogrammes per hectare"; given—

$$1 \text{ cwt.} = 50{\cdot}8024 \text{ kilos.}$$
$$1 \text{ acre} = {\cdot}40467 \text{ hectares.}$$

$$\begin{array}{r} \underline{125}{\cdot}\underline{5} \\ 40_{\backslash}4_{\backslash}6_{\backslash}7 \;)\; 5080240 \\ 10335 \\ 2242 \\ 219 \end{array}$$

Here it can be easily found that—

$$\text{R. E.} \prec {\cdot}00002,$$
$$\text{and } \therefore \text{ A. E.} \prec {\cdot}003.$$

Ex. (98) Find the side of a square whose diagonal is 100 ft.; given that $\sqrt{2} = 1{\cdot}414213\dot{6}$.

Ex. (99) Express "1 cub. ft. per sq. chain" in "litres per hectare"; given 1 cub. ft. = 28·31531 litres. 1 sq. chain = ·0928997 hectares.

Ex. (100) Express "100 kilogrammes per hectare" in "cwt. per acre"; given 1 kilog. = 2·2046 lbs.

Reciprocal Numbers.

63. If the product of two numbers is equal to unity, each is called the *reciprocal* of the other.

Thus 7 and $\frac{1}{7}$ are reciprocals, as are also $\frac{11}{9}$ and $\frac{9}{11}$.

A knowledge of the value of reciprocals will enable us to obtain *products by division*, and *quotients by multiplication*.

Thus, since $\frac{1}{273} = \cdot 00366$

if we have to find what space 8569 cubic inches of gas would occupy if it expanded by $\frac{1}{273}$ of its own bulk, we might work either by (i) division or (ii) by multiplication.

(i)		(ii)
	31·4	
27,3)	8569	8569
	379	25·707
	106	5·141
	8600·4	·514
		8600·4

The student should be able to avail himself of both substitutions; some calculators prefer to substitute multiplication for division, others division for multiplication; the advantage of the latter substitution is that "the figures of the result that are most wanted are soonest found".*

Note that, since $\frac{1}{273} = \cdot 00366$ app.,

$\therefore\ \frac{1}{366} = \cdot 00273$ app.

This last is approximately the fraction of itself, by which a mean solar day falls short of a sidereal day.

64. From the algebraical result,

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \frac{a^3}{x^4}, \text{ etc.,}$$

many reciprocals can be easily written down to a great many decimal places, thus, since $98 = 100 - 2$,

$\frac{1}{98} = \cdot 010204081632$

Similarly—

$\frac{1}{998} = \cdot 001002004008016032064128256$

$\frac{1}{9997} = \cdot 0001000300090027008102430729$

* De Morgan, *Examples of the Processes of Arithmetic and Algebra.*

Ex. (101) Write down $\frac{1}{97}$ as a decimal true to 7 places.
Ex. (102) Write down $\frac{1}{997}$ as a decimal true to 17 places.
Ex. (103) Write down $\frac{1}{9899}$ as a decimal true to 28 places.

Approximation in Vulgar Fractions.

Continued Fractions.

65. It frequently happens that a given vulgar fraction, even when in its lowest terms, has its numerator and denominator too large to allow of its use when rapidity of calculation or ease of physical application is more desired than minute accuracy. When this is the case, approximate values may be obtained, of any required degree of accuracy, by converting it into what is called a **continued fraction,** *i.e.*, a fraction of the form :—

$$a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{g + \text{etc.}}}}$$

In the form of continued fraction most used in numerical calculations $b = d = f = 1$. Thus it may be easily verified by actual reduction that—

$$3 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{2 + \frac{1}{6}}}} = \frac{967}{303}$$

$$\text{and } 3, 3\tfrac{1}{5}, 3 + \cfrac{1}{5 + \frac{1}{4}}, 3 + \cfrac{1}{5 + \cfrac{1}{4 + \frac{1}{2}}}$$

are successive approximations to $\frac{967}{303}$. The fractions $\frac{3}{1}$, $\frac{16}{5}$, $\frac{67}{21}$, $\frac{150}{47}$ obtained from these by simplification are called the first, second, third, fourth **convergents.**

The method of finding the successive numbers 3, 5, 4, 2, 6, by means of which these convergents are formed, is that of finding the G.C.M. of the numerator and denominator of the given vulgar fraction, of which it is required to find approximate values. Thus:—

303	967	3, 5, 4, 2, 6
13	58	
1	6	

3, 5, 4, 2, 6 are the successive quotients obtained by the successive divisions 967 by 303, 303 by the remainder 58, 58 by the remainder 13, 13 by the remainder 6, 6 by the remainder 1.

66. The successive convergents may be easily formed by the following rule, quoted in a concise form from Carr's *Synopsis of Elementary Results in Pure Mathematics.*

Write the quotients in a row, and the first two convergents (in the example $\frac{3}{1}$ and $\frac{16}{5}$) at sight.

Multiply the numerator of any convergent by the next quotient and add in the previous numerator. The result is the numerator of the next convergent.

Proceed in the same way to determine the denominator.

Thus, writing down the quotients—

$$3 \qquad 5 \qquad 4 \qquad 2 \qquad 6,$$

we obtain for convergents—

$$\frac{3}{1} \qquad \frac{16}{5} \qquad \frac{67}{21} \qquad \frac{150}{47} \qquad \frac{967}{303}.$$

Here $67 = 16 \times 4 + 3$ $\quad 150 = 67 \times 2 + 16$ $\quad 967 = 150 \times 6 + 67$

and $21 = 5 \times 4 + 1$ $\quad 47 = 21 \times 2 + 5$ $\quad 303 = 47 \times 6 + 21$

The last convergent is always the given fraction expressed in its lowest terms.

$$\text{Thus } \frac{6281}{326041} = \frac{1}{51+}\,\frac{1}{1+}\,\frac{1}{10}.$$ (See p. 17.)

$$= \frac{1}{51 + \frac{10}{11}},$$

$$= \frac{11}{571}.$$

The student should be able to see by ordinary arithmetic that the successive convergents are alternately in excess and in defect.

It is further shown in algebraical treatises that:—

(1) Each convergent is nearer to the true value than the convergent which precedes it.

(2) The difference between any two consecutive convergents is the reciprocal of the product of their denominators.

These statements may be verified to any extent by actual trial. Special attention should be given to (2), since it enables us to assign an upper limit to the error made.

Thus, $\frac{16}{5} - \frac{3}{1} = \frac{1}{1 \times 5}$; $\frac{16}{5} - \frac{67}{21} = \frac{1}{5 \times 21}$; $\frac{150}{47} - \frac{67}{21} = \frac{1}{21 \times 47}$; $\frac{150}{47} - \frac{967}{303} = \frac{1}{47 \times 303}$.

Now, as the true value of the fraction is intermediate in value between any two consecutive convergents, it follows that the error made in taking any convergent is less than the reciprocal of the product of its denominator and that of the next convergent.

Thus the "error in defect" of $\frac{67}{21} \prec \frac{1}{21 \times 47}$, and the "error in excess" of $\frac{150}{47} \prec \frac{1}{47 \times 303}$. For the sake of convenience the continued fraction—

$$a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{g}}}$$

is sometimes written—

$$a + \frac{b}{c +}\,\frac{d}{e +}\,\frac{f}{g}.$$

$$\text{Thus, } \frac{967}{303} = 3 + \frac{1}{5 +}\,\frac{1}{4 +}\,\frac{1}{2 +}\,\frac{1}{6}.$$

Another worked example is appended by which it appears that—

$$\frac{87969}{365256} = \frac{1}{4+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{5+}, \text{ etc.}$$

87969	365256	4, 6, 1, 1, 2, 1, 5.
7689	13380	
1998	5691	
303	1695	

Hence, writing down the quotients—

4 6 1 1 2 1 5,

we obtain for the successive convergents—

$$\tfrac{1}{4},\quad \tfrac{6}{25},\quad \tfrac{7}{29},\quad \tfrac{13}{54},\quad \tfrac{33}{137},\quad \tfrac{46}{191},\quad \tfrac{263}{1092}.$$

This last example has been chosen on account of an interesting astronomical application. For the sidereal revolutions of Mercury and the Earth take place respectively in 87·969 and 365·256 days. Consequently, if between two transits of Mercury across the Sun's disc, the planet makes m and the Earth n revolutions—

$$m\ 87{\cdot}969 = n\ 365{\cdot}256.$$

$$\frac{87969}{365256} = \frac{n}{m},$$

and we have to find what integral values of m and n will give the value of the fraction correctly enough; the convergents, $\frac{1}{4}$, $\frac{6}{25}$, are not sufficiently accurate, but transits may be expected to occur at the same node at intervals of 7, 13, 33, or 46 years. See Main's *Practical and Physical Astronomy*.

As further examples, consider the ratios—

$$\frac{\text{Circumference of circle}}{\text{Diameter}},\quad \frac{\text{Metre}}{\text{Yard}}.$$

The former, usually denoted by π, is to 9 significant figures, 3·14159265, which is easily found to be $3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}$.

Hence $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, . . . are successive approximations to the value of π.

Again—

$$\frac{\text{Metre}}{\text{Yard}} = \frac{39{\cdot}37079}{36} = \frac{3937079}{3600000} = 1 + \frac{1}{10+}\frac{1}{1+}\frac{1}{2+}.$$

Hence 1, $\frac{11}{10}$, $\frac{12}{11}$, $\frac{35}{32}$, are successive approximations. The third and fourth values may be given in more familiar language, thus:—

$$\left.\begin{array}{l}11 \text{ metres} = 12 \text{ yards}\\ 32 \text{ metres} = 35 \text{ yards}\end{array}\right\}\text{very nearly.}$$

67. In connection with the practical use of Continued Fractions in obtaining approximate values, the following theorem should be noted:—

If $\frac{a}{b}$ and $\frac{c}{d}$ be two given unequal fractions, and m and n any two positive integers, then the fraction $\frac{ma + nc}{mb + nd}$ lies between the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

For the proof, which is simple, the student is referred to Algebraical Treatises.

Hence, if $\frac{a}{b}$ and $\frac{c}{d}$ are both approximations to a true value and m and n any positive integers, $\frac{ma + nc}{mb + nd}$ is also an approximation, which will be in excess if each of the given fractions is in excess, and in defect if each of the given fractions is in defect, and which will in any case be a nearer approximation than one of the given fractions; though if the errors of the given fractions are of opposite sign we shall not know whether the new approximation is in excess or defect.

Thus from the given approximations in excess, $\frac{22}{7}$, $\frac{355}{113}$ to the value of π (see p. 75), we can obtain any number of new approximations of the form $\frac{22m + 355n}{7m + 113n}$, which will all be in excess.

But if from the approximations $\frac{22}{7}$, $\frac{333}{106}$, whose errors are of opposite sign, we obtained new approximations of the form $\dfrac{22m + 333n}{7m + 106n}$, all that we should know of any one of these, without further examination, would be that it was closer than $\frac{22}{7}$.

The theorem is sometimes of use in giving a new approximation with a more convenient numerator or denominator. Thus $\frac{22 + 355}{7 + 113}$ gives an approximation in excess, $\frac{377}{120}$, to the value of π, useful when the diameter was divided, as by the older writers on Trigonometry, into 120 equal parts.

Again we find in Lupton's *Numerical Tables and Constants* the length of the tropical year given as 365·24224 mean solar days, and calculating the first four convergents to ·24224 we find them to be $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$.

Now, since $\frac{1}{4}$ and $\frac{8}{33}$ are approximations in excess, $\frac{1 + 8 \times 12}{4 + 33 \times 12}$, *i.e.*, $\frac{97}{400}$, is also an approximation in excess. Hence the Gregorian Calendar, which intercalates 97 leap years in every 400 years, gives the year slightly too long, and not so accurate as the intercalation of 8 days in every 33 years, said to have been used by the Persians. A cycle of 400 years is, however, convenient.

68. A very interesting application of Continued Fractions to a practical problem may be found in a letter to the editor of the *Engineer* (10th April, 1891) by Prof. Unwin. The problem is that of finding a set of change wheels for cutting a millimetre screw, with very great accuracy, on an English lathe having a leading screw of eight threads to an inch.

Forming a series of convergents to the ratio required $\dfrac{25{\cdot}3995}{8}$, we get $\frac{3}{1}$, $\frac{16}{5}$, $\frac{19}{6}$, $\frac{54}{17}$, $\frac{73}{23}$, $\frac{127}{40}$, $\frac{1216}{383}$.

$\frac{19}{6}$ and $\frac{54}{17}$ easily give sets of change wheels, but are not very approximate. $\frac{127}{40}$ is very accurate, but the ordinary wheels in a set fail to give the ratio.

But $\frac{73+127}{23+40}$, *i.e.*, $\frac{200}{63}$, gives a ratio which is convenient and very approximate.

It may interest the student to find out how the following approximations, $\frac{203}{64}$, $\frac{238}{75}$, $\frac{92}{29}$, $\frac{35}{11}$, $\frac{51}{16}$ to the same ratio, may be obtained from the same set of convergents.

Ex. (104) Express as continued fractions—

(i) $\frac{1769}{5537}$, (ii) $\frac{59476}{13957}$, (iii) $\frac{55677644}{10000000}$, (iv) $\frac{30\cdot4368535}{20\cdot5305887}$,
(v) 3·14159265359.

Ex. (105) Find the first six convergents of each of the above continued fractions.

69. Continued Fractions sometimes arise in a way differing entirely from that described in § 65.

For instance, since the integral part of $\sqrt{19}$ is 4, the fractional part satisfies the equation $(x + 4)^2 = 19$.

$$x^2 + 8x = 3$$

$$x = \frac{3}{8+x}$$

$$= \frac{3}{8+}\,\frac{3}{8+}\,\frac{3}{8+}, \text{ etc., } \textit{ad infin.}$$

Again, if a straight line of unit length is divided medially (*i.e.*, as Euclid II. 11) we see that the major segment satisfies the equation—

$$x^2 = 1 - x$$

$$x^2 + x = 1$$

$$x = \frac{1}{1+x}$$

$$= \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}.$$

Here the successive convergents—

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21},$$

are formed rapidly.

$\frac{13}{21}$ is the one given by Lord Grimthorpe, in a letter to the *English Mechanic*, as useful in the practical construction of approximately regular pentagons in architecture, 13 being taken for a side and 21 for the diagonal.

70. In connection with the application of Vulgar Fractions to abbreviated work, the attention of the student is again drawn to the importance of the use of the methods of "Practice" for affecting reductions and approximations in preference to direct multiplication and division, and to the convenience of using vulgar in conjunction with decimal fractions (see p. 24).

The simplest of all cases is that in which we have to multiply by a mixed number like $3\frac{1}{7}$. The author has found great difficulty in restraining pupils from converting the multiplier either into an improper fraction or a decimal before multiplying; naturally we ought first to use the integer (here 3) as a multiplier, and then add on the required fractional part (here $\frac{1}{7}$) of the multiplicand.

As examples, take the products, each to 5 significant figures, of (i) 5·8321 and $3\frac{1}{7}$, (ii) 43·692 and $4\frac{7}{9}$.

$$\text{(i)}\quad \begin{array}{r} \underline{5{\cdot}8321} \\ 17{\cdot}4963 \\ \underline{{\cdot}8332} \\ 18{\cdot}329 \end{array} \qquad\qquad \text{(ii)}\quad \begin{array}{l} \underline{43{\cdot}692} \\ 174{\cdot}768 \\ \underline{\;\;\;4{\cdot}854\tfrac{2}{3} \times 7} \\ 208{\cdot}75 \end{array}$$

71. As to the advantage of using vulgar fractions for *operators* in preference to decimals, see an article by Prof. A. Lodge in the *Mathematical Gazette* for April, 1894, on *Approximations and Reductions*.

If a given decimal multiplier is likely to be of frequent use, it is worth while trying to invent some equivalent operator consisting of a set of simple vulgar fractions. Take, for instance, the case of finding the linear dimensions of a solid which is to be similar to a given solid, but of double its capacity. Each of its linear dimensions will be found by multiplying the corresponding one of the given solid by $\sqrt[3]{2}$. Now $\sqrt[3]{2} = 1{\cdot}2599$ . . . Hence to a high degree of accuracy we may take for our operator $1 + \frac{1}{4} + \frac{1}{100}$ ($= 1{\cdot}26$). As a

particular instance we find the diameter of a sphere, which shall have a volume double that of a sphere whose diameter is 29 ft., to be nearly 36·54 ft.

$$\begin{array}{r} 29 \\ 7{\cdot}25 \\ {\cdot}29 \\ \hline 36{\cdot}54 \end{array}$$

The following are taken from Prof. Lodge's article :—

The commonly used approximation, $3\frac{1}{7}$, for the value of π is true only to *three* significant figures, but (i) $3\frac{1}{7} - \frac{1}{800}$ is true to *five* significant figures, and thus yields a second approximation by a correction of the first. The value (ii) $3 + \frac{1}{8} + \frac{1}{60}$, which is slightly higher, may be used as a check.

$$3 + \tfrac{1}{7} - \tfrac{1}{800} = 3{\cdot}1416\tfrac{1}{14}\,;\ 3 + \tfrac{1}{8} + \tfrac{1}{60} = 3{\cdot}1416\tfrac{2}{3}.$$

Two other expressions, each equal to 3·1416, are worthy of notice.

(iii) $(3 + \frac{1}{7})\,(1 - {\cdot}0004)$; (iv) $3(1 + \frac{1}{20})\,(1 - \frac{1}{400} - \frac{1}{6000})$.

The advantage of (iii) is that if the product of several factors including π be required, the approximation $3\frac{1}{7}$ can be used first, and at the end of the multiplication the correction ($\frac{4}{10000}$ of the product) subtracted. The advantage of (iv) lies in its containing 3 as a factor, which is useful in the reduction of degrees to circular measure. To illustrate their use we proceed *to find the circumference of a circle when the diameter is* 37·158.

$$\begin{array}{cccc}
\text{(i)} & \text{(ii)} & \text{(iii)} & \text{(iv)} \\
\begin{array}{r} 37{\cdot}158 \\ \hline 111{\cdot}474 \\ 5{\cdot}308 \\ \hline 116{\cdot}782 \\ {\cdot}046 \\ \hline 116{\cdot}736 \\ \\ \end{array} &
\begin{array}{r} 37{\cdot}158 \\ \hline 111{\cdot}474 \\ 4{\cdot}645 \\ {\cdot}619 \\ \hline 116{\cdot}738 \\ \\ \\ \end{array} &
\begin{array}{r} 37{\cdot}158 \\ \hline 111{\cdot}474 \\ 5{\cdot}308 \\ \hline 116{\cdot}782 \\ {\cdot}047 \\ \hline 116{\cdot}735 \\ \\ \end{array} &
\begin{array}{r} 37{\cdot}158 \\ 1{\cdot}8579 \\ \hline 39{\cdot}0159 \\ \hline 117{\cdot}0477 \\ \hline {\cdot}2926 \\ {\cdot}0195 \\ \hline 116{\cdot}7356 \end{array}
\end{array}$$

The reciprocal of π to eight significant figures is ·31830989; To this we have the following approximations:—

(i) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200}) = \cdot 3183\frac{1}{3}$.
(ii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200} + \frac{1}{50000}) = \cdot 31831\frac{1}{3}$.
(iii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200} + \frac{1}{40000}) = \cdot 318308\frac{1}{3}$.
(iv) $\frac{1}{3} (1 - \frac{1}{22}) (1 + \cdot 0004) = \cdot 318309\frac{1}{11}$.

As an illustration *we find the diameter of a circle whose circumference is* 116·74

(i)	(ii)	(iii)	(iv)
116·74	116·74	116·74	116·74
38·913	38·913	38·913	22) 38·913
1·167	1·167	1·167	6·59
·584	·584	·584	11·1
37·162	·002	·003	1·769
	37·160	37·159	37·144
			·015
			37·159

The multiplier for converting circular measure into degrees to seven significant figures is 57·29578, to which we have the approximation $60 (1 - \frac{1}{20} + \frac{1}{200}) = 57\cdot 30$, true to four significant figures.

There are cases in which a decimal approximation is the best. Thus to five significant figures—

$$\pi^2 = 10 (1 - \cdot 01304) \quad \frac{1}{\pi^2} = \frac{1}{10} (1 + \cdot 01321).$$

Again, since 1 metre = 39·370 inches, we have approximately 1 metre = $39\frac{3}{8}$ inches. This may be remembered easily in two forms—

1 metre = 3 ft. $3\frac{3}{8}$ in.
32 metres = 35 yds.

Hence—

To reduce metres to yards multiply by $1 + \frac{1}{12} + \frac{1}{98}$.

„ „ „ feet „ „ $3 + \frac{1}{4} + \frac{1}{32}$.

„ cmm. „ inches „ „ $\frac{4}{10} - (\frac{1}{200} + \frac{1}{800})$.

„ yards „ metres „ „ $1 - \frac{1}{10} + \frac{1}{70}$.

„ feet „ „ „ „ $\frac{3}{10} + \frac{1}{200} - \frac{1}{5000}$.

„ inches „ cmm. „ „ $\frac{10}{4} + \frac{4}{100}$.

As an example we *reduce* (i) 5·6342 *metres to yards;* (ii) 4·378 *cmm. to ins., and check in each case by reduction to the original denomination.*

(i)		(ii)	
5·6342	6·1624	4·378	1·724
·46952	·61624	1·7512	4·310
·05869	5·54616	·0219	68·96
6·1624	·08803	·0055	4·379
	5·6342	1·724	

A table will be added of these and other formulæ of reduction.

Approximation in Involution.

72. If involution be treated as ordinary multiplication it may be abbreviated by the process described on p. 61. But *Horner's Method*, which the student has already been strongly recommended to adopt, is also capable of considerable abbreviation as exemplified in the following sections, which should present little difficulty *if the corresponding sections of Part I. have been mastered.*

Approximate Squares.

Suppose it were required to find the first four figures in the square of 83792, we might cut short the process, which is given in full on p. 26, after the third line thus :—

8	6400
163	688900
1667	700569
16‚7‚4	702076
	702109

and the first four figures are found to be 7021.

Note that if we were asked for the square of 83792, *true to 4 significant figures*, we should have to reply 7021000000.

The student should compare the approximate with the complete work.

As an additional example, we will find the square of 956436 true to 5 significant figures.

9	8100
185	902500
1906	913936
1‚9‚1‚2	914701
	914758
	914769

Ans. 914770000000.

Ex. (106) Find the first 4 figures of the square of 837925.

Ex. (107) Find the square of 33793 to 4 significant figures.

Ex. (108) Find the first 5 figures of the square of 956437.

Ex. (109) Find the square to 5 significant figures of (i) 956436·7; (ii) 83·782; (iii) 847·92.

Ex. (110) Given that the first 5 figures of the 200th power of 2 are 12676, find the first 4 figures of the 400th power.

73. The approximate evaluation of a *quadratic expression* can be effected by a slight modification of the process for approximate *squaring*.

(i) Evaluation of $x^2 + 3x + 5$, when $x = 1{\cdot}375$, correct to units.

1	3	5
	4	9
	5	10·59
	5·3	10·98
	5·6	11·01
	Result.	11.

Ex. (111) Evaluate—

$$\left.\begin{array}{r} x^2 + 3x + 5 \text{ (i) when } x = 2{\cdot}468 \\ \text{(ii) when } x = 3{\cdot}579 \end{array}\right\} \text{correct to units.}$$

(ii) Evaluation of $5x^2 + 6x$, when $x = 3{\cdot}7951$, to 4 significant figures.

5	6	0
	21	63
	36	90·65
	39·5	94·5605
	43·0	94·7800
	43·45	94·7844
	43·90	
	Result.	94·78.

Ex. (112) Evaluate $5x^2 + 6x$, when $x = 3{\cdot}7951$, correct to units.

Ex. (113) Evaluate $7x^2 + 3x$, when $x = 2{\cdot}4683$, to 4 significant figures.

(iii) Evaluation of $4617x^2 + 29318x$, when $x = 21{\cdot}75$, to 5 significant figures.

4617	29318	0
	121658	2433160
	213998	2651775
	218615	2810297
	223232	2821792
	22646	
	22969	
	2299	
	Result.	2821800.

Compare the approximate with the complete work (p. 29). Note that when the coefficient of x^2 is of more than one digit we cut it short two places at a time, whenever we cut the column short by one figure.

Ex. (114) Evaluate $4617x^2 + 29318x$, when $x = 21{\cdot}86$, to 5 significant figures.

(iv) Evaluation of $561983x^2 + 28617x + 493$, when $x = 3{\cdot}4287$, to units.

56 19 83	28617	493
	1714566	5144191
	3400515	6594314·28
	3625308·2	6671541·10
	3850101·4	6702557·74
	3861341	6705275·14
	3872581	
	387708	
	388158	
	38820	

Result. 6705275.

Ex. (115) Evaluate to units:—

(i) $4617x^2 + 29318x$ when $x = 31{\cdot}75$.

(ii) $5943x^2 + 54761x$ when $x = 1{\cdot}473$.

(iii) $561983x^2 + 28617x + 493$ when $x = 3{\cdot}4287$.

(iv) $742356x^2 + {\cdot}539x$ when $x = 31{\cdot}75$, $x = 1{\cdot}473$, and when $x = 3{\cdot}4287$.

Approximate Cubes.

74. Suppose it were required to find the first four figures in the cube of 83792 we might cut short the process, which is given in full on p. 32, after the third line, thus:—

8	64	512000
16	19200	571787
243	19929	586375
246	2066,7	588266
2,49	2084	588308
	21,0,1	

and the first four figures are found to be 5883.

Note that if we were asked for the cube of 83792 *true to 4 significant figures* we should have to reply 588300000000000, or 5883×10^{11}.

To make the method of abbreviation more apparent, the work for finding the same cube true to 7 significant figures is given below.

8	64	512000
16	19200	571787000
243	19929	586376253
246	2066700	588269817
2497	2084179	588311943
,25,04	210170,7	
	210396	
	21062,1	
	21063	

Result. 588311900000000.

The student should, in each case, compare the approximate with the complete work, and note that when the middle column is cut short by one figure the left column is simultaneously cut short by two figures.

Ex. (116) Find the first 4 figures of the cube of 987·5.

Ex. (117) Find the cube of 98·65 true to 4 significant figures.

Ex. (118) Find the cubes of 9·8753 and 9·87534 true to 7 significant figures.

75. The approximate evaluation of a *cubic expression* can be effected by a slight modification of the process for approximate *cubing*.

(i) Evaluation of $x^3 + 3x^2 + 8x + 4$, when $x = 5{\cdot}673$, to 4 significant figures.

1	3	8	4
	8	48	244
	13	113	318·496
	18·6	124·16	328·086
	19·2	135·68	328·497
	19·8	137·0	

Result. 328·5.

Ex. (119) Evaluate—

$x^3 + 3x^2 + 8x + 4$ (i) when $x = 5{\cdot}684$,
(ii) when $x = 4{\cdot}673$.

(ii) Evaluation of $5x^3 + 6x^2$, when $x = 3{\cdot}7951$, to 7 significant figures.

5	6	0	0
	21	63	189
	36	171	335·405
	51	209·15	358·384295
	54·5	249·75	359·690575
	58·0	255·3255	359·716733
	61·5	260·9415	
	61·95	261·256	
	62·40	261·570	
	62·85	261·58	

Result. 359·7167.

Ex. (120) Evaluate $5x^3 + 6x^2$, when $x = 3{\cdot}7951$, to 4 significant figures.

Ex. (121) Evaluate $4x^3 + 5x^2$, when $x = 3{\cdot}7951$, (i) to 4, (ii) to 7 significant figures.

Evaluation of Series.

76. The chief point in the evaluation of series is to render the work *continuous*, so that successive numbers may, as far

as possible, be derived from those immediately preceding them.

As an example take the evaluation of π by means of the Trigonometrical series.

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}$$

$$\text{Putting } x = \frac{1}{\sqrt{3}} \text{ we get } \tan^{-1} x = 30^{0} = \frac{\pi}{6}.$$

$$\text{Hence } \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \text{etc.}$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}.3.3} + \frac{1}{\sqrt{3}.5.3^2} - \frac{1}{\sqrt{3}.7.3^3} + \text{etc.}$$

$$\pi = \sqrt{12} - \tfrac{1}{3}.\ \frac{\sqrt{12}}{3} + \tfrac{1}{5}\ \frac{\sqrt{12}}{9} - \tfrac{1}{7}.\ \frac{\sqrt{12}}{3.9} + \text{etc.}$$

$$= \sqrt{12} + \tfrac{1}{5}\frac{\sqrt{12}}{9} + \tfrac{1}{9}.\ \frac{\sqrt{12}}{9.9} + \tfrac{1}{13}.\ \frac{\sqrt{12}}{9.9.9} + \text{etc.}$$

$$- \left(\tfrac{1}{3}.\ \frac{\sqrt{12}}{3} + \tfrac{1}{7}.\ \frac{\sqrt{12}}{3.9} + \tfrac{1}{11}.\ \frac{\sqrt{12}}{3.9.9} + \text{etc.}\right)$$

Here, although the actual terms in each of the above series cannot be obtained continuously, each of the two sets of numbers—

$$\sqrt{12},\ \frac{\sqrt{12}}{9},\ \frac{\sqrt{12}}{9.9},\ \frac{\sqrt{12}}{9.9.9},$$

$$\frac{\sqrt{12}}{3},\ \frac{\sqrt{12}}{3.9},\ \frac{\sqrt{12}}{3.9.9.},\ \frac{\sqrt{12}}{3.9.9.9}.$$

can be. It will therefore be convenient to obtain these sets of numbers first, and take the required parts of them afterwards. In the annexed work a space was left beneath each of the successive numbers for the subsequent insertion of the actual terms of the series.

	3·4641016	1·1547005
	·3849002	**·3849002**
	·0769800	·1283001
	·0427667	**·0183286**
	·0047519	·0142556
	·0047519	**·0012960**
	·0003655	·0015839
	·0005280	**·0001056**
	·0000311	·0001760
	·0000587	**·0000093**
	·0000028	·0000196
	·0000065	**·0000009**
	·0000003	**·4046406**
	3·5462332	
	·4046406	
π =	**3·1415926**	

In manuscript work the actual terms of the series here printed in thick type might be written *in red ink*, so that no confusion should be possible between them and the alternate numbers which have to be omitted in adding up.

Ex. (122) Evaluate the series given above (i) when $x = \frac{1}{2}$, (ii) $x = \frac{1}{3}$.

Approximation in Evolution.

Square Root.

77. When $\overline{n+1}$ digits of a square root have been found by the ordinary process, n more (sometimes $\overline{n+1}$ more) may be found by a division process only.

We take as an example the extraction of the square root of 2990667969, and first give the ordinary process in full, merely using the "Italian method" (see pp. 16, 17) for shortening the work.

$$\begin{array}{lr} & \underline{54687} \\ & 2990667969 \\ 104 & 490 \\ 1086 & 7466 \\ 10928 & 95079 \\ 109367 & 765569 \\ & \cdots \end{array}$$

Having thus obtained the square root (54687) by the ordinary process, we now proceed to show that after we had obtained the first three digits (5, 4, 6) the next two (8, 7) might have been obtained by division, thus :—

$$\begin{array}{lr} & \underline{54687} \\ & 2990667969 \\ 104 & 490 \\ 1086 & 7466 \\ 1092 & 9500 \\ & 7640 \end{array}$$

Here the remainder, after obtaining the digit 6 by the ordinary process, being 950, instead of bringing down the next group (79) of two figures, we merely append a 0 to the 950 and divide the 9500 by 1092 (*i.e.*, double of the 546 already obtained); we obtain quotient 8, which we put in the root with remainder 764. Appending another 0 we see that the next digit is 7.

If the first of the $\overline{n+1}$ digits of the root extracted by the ordinary process be $\geqq 5$ we can obtain $\overline{n+1}$ more by division; if < 5, n only can be depended on.

78. The extraction may be still further abbreviated by using the method of abbreviated division in the latter part of it (see pp 66, 67), *i.e.*, by cutting short the divisor 1092, instead of appending noughts to the remainders, thus :—

	$\overline{54687}$
	2990667969
104	490
1086	7466
1092	950
	76

The extraction of $\sqrt{2}$ to 9 significant figures is given below as a further example of the same method.

	$\overline{1{\cdot}41421356}$
	2
24	100
281	400
2824	11900
28282	60400
28284	3836
	1008
	160
	19

When the division process is thus abbreviated the last figure (here 6) is doubtful. To save the calculation of the possible amount of error (as on pp. 68, 69) it is a good plan to extract the root to 1 or 2 places more than are required. The student's attention is drawn to a curious exceptional case which sometimes occurs. Suppose, for instance, we begin to extract $\sqrt{32{\cdot}48}$.

	$\overline{5{\cdot}6}$
	32·48
106	7·48
112	1·12

Here the divisor appears to go once into the dividend without either appending a 0 or abbreviating the divisor; in this case append nines to the root, as many in number (in this case 2) as the figures that would have been obtained by

the division. Hence $\sqrt{32 \cdot 48} = 5 \cdot 699$. (Ruchonnet's *Éléments de Calcul Approximatif.*)

Ex. (123) Find to 12 significant figures the square roots of 2, 3, 28, 31, 44, 67.

79. A curious method of approximating to the square root of a number is given by Hutton in his *Miscellanea Mathematica.*

Let N be the number (supposed to be an integer, for simplicity) whose square root it is desired to extract, and let p and q be other integers, found by trial such that—

$$p^2 = q^2 N + 1,$$

then $\frac{p}{q}$ is an approximate value; and if a series of fractions—

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$$

be formed by the process—

$$\frac{p_1}{q_1} = \frac{p}{q} - \frac{1}{2pq}$$
$$\frac{p_2}{q_2} = \frac{p_1}{q_1} - \frac{1}{2p_1 q_1}$$

each of these is a closer approximation to the value of $\sqrt{N}$ than the preceding one. For it can be easily shown that—

$$p_1^2 = q_1^2 N + 1$$
$$p_2^2 = q_2^2 N + 1$$

and so on.

Hence each is in excess of the true value. Also in taking $\frac{p}{q}$ as a value of $\sqrt{N}$ it can be shown that the error in excess is $> \frac{1}{2pq}$ but $< \frac{1}{2aq^2}$ where a is the integral part of $\sqrt{N}$.

Take $\sqrt{2}$ as an example.

By trial $3^2 = (2^2 \times 2) + 1$.

$$\tfrac{3}{2} - \tfrac{1}{2.2.3} = \tfrac{3}{2} - \tfrac{1}{12}$$
$$= \tfrac{17}{12} = 1{\cdot}4166\ ..$$
$$\tfrac{17}{12} - \tfrac{1}{2.12.17} = \tfrac{17}{12} - \tfrac{1}{408}$$
$$= \tfrac{577}{408} = 1{\cdot}414215\ ..$$
$$\tfrac{577}{408} - \tfrac{1}{2 \times 408 \times 577} = \tfrac{577}{408} - \tfrac{1}{470832}$$
$$= \tfrac{665857}{470832} = 1{\cdot}414213562375.$$

Here the error is known to be less than $\frac{1}{2(470832)^2}$, and as a matter of fact the value found only differs from the true value of $\sqrt{2}$ in the last digit, which should be 3 instead of 5.

As another example, take N = 920.

By trial $91^2 = (3^2 \times 920) + 1$.

$$\tfrac{91}{3} - \tfrac{1}{2.3.91} = \tfrac{16561}{546}$$
$$= 30{\cdot}33150183$$

Here the error is known $< \frac{1}{60(546)^2}$, and the true value to 12 significant figures is 30·3315017762.

There is, however, no advantage in this method over the one previously given; but as it affords good exercise to the student in calculating beforehand the number of places which he may rely on, and in comparing the results thus obtained with those derived from the other, a few numerical identities are added from Meyer Hirsch's *Sammlung von Beispielen, Formeln und Aufgaben* which yield the first approximations to $\sqrt{28}$, $\sqrt{31}$, $\sqrt{44}$, $\sqrt{67}$.

$$127^2 = 24^2 \times 28 + 1$$
$$1520^2 = 273^2 \times 31 + 1$$
$$199^2 = 30^2 \times 44 + 1$$
$$48842^2 = 5967^2 \times 67 + 1$$

Similarly, if $p^2 = q^2N - 1$

$\frac{p}{q} + \frac{1}{2pq}$ is a closer approximation to $\sqrt{N}$ than $\frac{p}{q}$.

80. The method of solving quadratics described on p. 36 can be abbreviated in a similar way. Thus having given the equation :—

$$x^2 + 3x = 7$$

we notice that—

$$\text{when } x = 1,\ x^2 + 3x = 4$$
$$\text{when } x = 2,\ x^2 + 3x = 10$$

and hence that a real positive root lies between 1 and 2; the root can be found to any required degree of accuracy as follows :—

$$
\begin{array}{lll}
 & & \underline{1{\cdot}5413813} \\
1 & 3 & 7 \\
 & \underline{1} & \underline{4} \\
 & 4 & 300 \\
 & \underline{1} & \underline{275} \\
 & 55 & 2500 \\
 & \underline{5} & \underline{2416} \\
 & 604 & 8400 \\
 & \underline{4} & \underline{6081} \\
 & 6081 & 231900 \\
 & \underline{1} & \underline{182469} \\
 & 60823 & 494310 \\
 & \underline{3} & \underline{486608} \\
 & 60826 & 77020 \\
 & & \underline{60826} \\
 & & 161940
\end{array}
$$

This should be abbreviated thus :—

$$
\begin{array}{lll}
 & & \underline{1{\cdot}5413813} \\
1 & 3 & 7 \\
 & 4 & 300 \\
 & 55 & 2500 \\
 & 604 & 8400 \\
 & 6081 & 231900 \\
 & 60823 & 49431 \\
 & 60826 & 770 \\
 & & 162
\end{array}
$$

As an additional example let us find the positive root of $x^2 - 5x = 11$, which by trial lies between 6 and 7.

$$\begin{array}{rll}
 & & \underline{6{\cdot}6533119} \\
1 & -5 & 11 \\
 & 1 & \ \ 500 \\
 & 76 & \ \ \ 4400 \\
 & 825 & \ \ \ \ 27500 \\
 & 8303 & \ \ \ \ \ 259100 \\
 & 83063 & \ \ \ \ \ \ \ 9911 \\
 & 83066 & \ \ \ \ \ \ \ 1604 \\
 & & \ \ \ \ \ \ \ \ \ 773
\end{array}$$

N.B. If a is a root of the equation—

$$ax^2 + bx + c = 0$$
$$-\ a \text{ is a root of}$$
$$ax^2 - bx + c = 0$$

Hence the above method may be used for finding negative roots; in fact, $-6{\cdot}6533119$ is a negative root of $x^2 + 5x = 11$.

Also, since the roots of the quadratic equation—

$$ax^2 + bx + c = 0$$

are known to be $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the student may make any number of examples which he can solve by the method described, and verify by the formula: thus $\dfrac{\sqrt{37} - 3}{2} = 1{\cdot}5413812$

$\dfrac{5 + \sqrt{69}}{2} = 6.6533119.$

81. The method is important as a special case of a general one by which Horner rendered it easy to find approximate real roots of equations of any degree, supposing such to exist, except in some unusual cases with which the student is unlikely to be troubled.

In the examples just solved, Horner's method has no superiority over that commonly used for solving quadratics. But it may be used with advantage, even for quadratics when the coefficients are large numbers.

Ex. Find a positive value of x to 4 significant figures which will satisfy the equation—

$$4617x^2 + 29318x = 2821800.$$

			21·75
4617	29318	2821800	
	121658	388640	
	213998	170025	
	218615	11503	
	223232		
	22646		
	22969		
	2299		

The student should compare the *direct* (p. 37) with the *inverse* process.

Ex. (124) Solve approximately the following quadratic equations (4 sig. figs.):—

$$\text{(i) } x^2 + 3x = 6\,; \text{ (ii) } 5x^2 + 6x = 95\,;$$
$$\text{(iii) } 4617x^2 + 29318x = 5123400\,;$$
$$\text{(iv) } 561983x^2 + 28617x = 6705275.$$

82. Quadratic equations may also be solved approximately by means of Trigonometrical Tables. (See p. 143.)

Cube Root.

83. The extent to which the cube root process may be cut short is best illustrated by a few numerical examples.

(i) Extraction of the cube root of 13 to 4 significant figures.

		2·351
2	4	13
4	12	5·000
6·3	13·89	·833
6·6	15·87	·023
6·9	16·2	

(ii) Extraction of the cube root of 13 to 6 significant figures.

		2·351335
2	4	13
4	12	5·000
6·3	13·89	·833
6·6	15·87	·022125
6·9	16·2175	·005558
6·95	16·5600	·000588
7·00	16·567	91
7·05		

Result. 2·35134.

Ex. (125) Extract the cube root of 14 (i) to 4, (ii) to 6, (iii.) to 9 significant figures.

(iii) Extraction of the cube root of 9 to 14 significant figures.

		2·08008382305190
2	4	9
4	12·0000	1·000000
6·08	12·4864	·001088000000000
6·16	12·9792000000	·000049624063488
6·24008	12·9796992064	·000010683412608
6·24016	12·9801984192	·000000299219388
6·24024	12·980217140	·000000039614468
	12·980235861	·000000000673730
	12·98024085	·000000000024718
	12·98024584	·000000000011738
	12·9802460	·000000000000056
	12·9802461	

Result. 2·0800838230519.

Ex. (126) Extract cube root of 9 (i) to 4, (ii) to 6, (iii) to 9 significant figures.

84. When the process has become familiar some of the figures here retained might be dispensed with, *e.g.*, the 12980 need not be repeated in the middle column, and the cyphers might be omitted from the third, as in the following example from De Morgan, in which the cube root of—

696536483318640035073641037

is found to 14 significant figures.

886437165·39328.

8	64	696536483318640035073641037
16	19200	184536
248	21184	15064483
256	2323200	1030027318
2646	2339076	87606774640
2652	235498800	16890934933
26584	235605136	389883585
26588	23571148800	154152392
265923	23571946569	12713651
265926	23572744347	927088
265929	23572930497	219894
	3116647	7736
	311930	664
	312196	193
	31235	5
	235731251	

Ex. (127) Extract the cube root of 696·536483318640 (i) to 4, (ii) to 6, (iii) to 9, (iv) to 12 significant figures.

85. The method of solving cubic equations described on p. 41 can be abbreviated in a similar way.

(i) Solution of $x^3 + 2x^2 + 3x = 52$ to 5 significant figures.

		2·95177
2	3	52
4	11	30·000
6	23·00	2·091000
8·9	31·01	72625
9·8	39·83	31707
10·75	40·3675	3049
10·80	40·9075	183
10·85	40·918	
	40·929	
	40·94	

Result. $x = 2·9518$.

Ex. (128) Find a value of x to 6 significant figures, satisfying $x^3 + x^2 + x = 90$.

(ii) Solution of $x^3 - 11x^2 + 5x = 14$ to 5 significant figures.

			10·6540
1	− 11	5	14
	− 1	− 5	64·000
	9	85	5·944
	19·6	96·76	·449
	20·2	108·88	5
	20·8	109·9	
		1110	

Result. $x = 10·654$.

Ex. (129) Find value of x to 5 significant figures, satisfying—

(i) $x^3 - 15x^2 + 63x = 50$.
(ii) $x^3 - 12x^2 + 45x = 53$.
(iii) $x^3 - 12x^2 + 57x = 94$.
(iv) $x^3 - 12x = 28$.
(v) $x^3 + 2x^2 - 23x = 70$.

86. For a complete account of Horner's method and its application to the solution of equations of any degree, the student is referred to the *Penny Cyclopædia* "Involution". Cubic equations may also be solved approximately by means of Trigonometrical Tables. See p. 144.

The solution of the equation $x^4 + 8x^2 + 16x = 440$ to 4 significant figures is given below to illustrate the ease with which it can be applied.

				3·9760
1	0	8	16	440
	3	17	67	239·0000
	6	35	172	24·5759
	9	62·00	238·249	1·9883
	12·9	73·61	315·676	98
	13·8	86·03	322·68	
	14·7	99·26	329·75	
	15·6	100		

$$x = 3{\cdot}976.$$

87. Equations of the form $ax = b + q$ in which a and b are constants, and q a quantity which depends on x. These frequently occur in the investigation of physical problems, and the following method may sometimes be advantageously employed for their numerical solution, especially when x is known to lie between easily ascertained limits, and q does not change rapidly with x.

(i) Find some rough approximation, x_1, to the value of x. If q is small compared with b this may be found by neglecting q and taking $x = \dfrac{b}{a}$.

(ii) Find the value q_1 of q when x_1 is put for x, and substitute it for q in the given equation.

This gives $x = \dfrac{b + q_1}{a}$ for a second approximation.

(iii) Find the value q_2 of q when $\frac{b + q_1}{a}$ is put for x, and substitute it in the given equation. This gives $x = \frac{b + q_2}{a}$ for a third approximation. The process to be continued until the required degree of accuracy is found.

The process is much facilitated if the successive values of q can be obtained from a set of mathematical tables.

Ex. Find approximately the positive root of the equation—

$$x^2 + 1000x = 9193.$$

Here the required root clearly lies between 9 and 10, and x^2 is small compared with $1000x$. Neglecting it, we obtain $x = 9{\cdot}193$.

By the tables, to 2 places of decimals, $(9{\cdot}193)^2 = 84{\cdot}46$.

Substituting this for x^2 we obtain $x = 9{\cdot}109$.

By the tables, to 2 places of decimals, $(9{\cdot}109)^2 = 82{\cdot}97$.

Substituting this for x^2 we obtain $x = 9{\cdot}110$.

The work might be arranged thus:—

$$\begin{array}{rcr} 9{\cdot}193 & & 9{\cdot}193 \\ \underline{{\cdot}08446} & \rightarrow & \underline{{\cdot}08297} \\ 9{\cdot}109 & & 9{\cdot}110 \end{array}$$

This is verified by proceeding to the next approximation, for we find on consulting the tables that $(9{\cdot}110)^2 = 82{\cdot}99$. . . .

Ex. (130) Solve the equation—

$$x^2 + 1000x = 8190.$$

88. q may involve other functions of x than its powers. It may, for instance, involve Trigonometrical, Logarithmic, or Exponential functions. For an example of the above method, in which logarithms are involved, see p. 122.

III. LOGARITHMS.

89.

1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16

Form a column of the numbers 1, 2, 4, 8, etc., by writing down unity and then doubling each number in succession to get the next. Opposite each of these *powers of* 2, and on the right-hand side of it, write down the number which shows what power of 2 it is, thus forming another column of the numbers 0, 1, 2, 3, etc. A little consideration will show that when we have once formed such a pair of columns we can use it to solve a variety of questions in Arithmetic, with little work beyond a mere inspection of the columns; *e.g.*—

(i) What is the product of 64 and 256?
Answer. 16384 (since $6 + 8 = 14$).

(ii) What is the quotient of 4096 by 32?
Answer. 128 (since $12 - 5 = 7$).

(iii) What is the square of 128?
Answer. 16384 (since $7 \times 2 = 14$).

(iv) What is the square root of 65536?
Answer. 256 (since $16 \div 2 = 8$).

(v) What is the fifth power of 8?
Answer. 32768 (since $3 \times 5 = 15$).

(vi) What is the seventh root of 16384?
Answer. 4 (since $14 \div 7 = 2$).

Having noticed that the calculation of the *products*, *quotients*, *powers* and *roots* of the numbers in the left column can be effected by *much simpler operations* with the *much smaller numbers* in the right column, an intelligent student, especially if acquainted with *fractional indices*, may easily be led to inquire whether some way does not exist of making the table complete by the insertion of numbers to fill up the gaps in the columns. A very slight acquaintance with

indices, possibly the mere knowledge that $a^{\frac{m+n}{2}} = \sqrt{a^{m+n}} = \sqrt{a^m \times a^n}$, may serve to show him that it is in his power to make the table as complete as he chooses, by the continued insertion between any two numbers in the left column of the *geometrical mean between them* (*i.e.*, of the square root of their product), accompanied by the insertion in the right column of the arithmetical mean (*i.e.*, of half the sum) of the two numbers opposite to them.

Thus between 2 and 4 in the left column insert 2·8284 ($\sqrt{2 \times 4}$) and write 1·5 ($= \frac{1+2}{2}$) opposite to it thus:—

2	1
2·8284	1·5
4	2

If he has mastered these considerations he has, in fact, ascertained for himself the possibility of constructing a table of **logarithms,** as the index numbers 0, 1, 2, 3 in the right column are called. Such tables have been in existence for more than 200 years, "during which time the labour of computation has been reduced for the mathematician to about a tenth part of its previous expense of time and trouble. The mathematical computer, however, has had them all to himself; they have never been applied, except indirectly, to the purposes of commerce or the wants of life." *

Napier, the inventor of Logarithms, constructed his tables by a method entirely different to the one suggested above, though that appears to have occurred to him afterwards. Another simple method, also due to Napier, may be found in Hutton's *Mathematical Tracts*, vol. i., pp. 369-372, or in the reprint of Napier's *Construction of the Wonderful Canon of Logarithms* (pp. 53, 62) (Blackwood).†

* De Morgan, *On the Use of Small Tables of Logarithms in Commercial Calculations.*

† These two works may also be consulted with advantage by those interested in the *history* of Logarithms.

90. Most of the more expeditious methods of calculating logarithms depend on theorems which are not of an elementary character. For an account of these the student is referred to one of the larger treatises on Algebra by Todhunter, Hall and Knight, or Smith.

The smaller treatises by the same writers supply proofs of the theorems on which the rules for practical work depend, and which the author is contented merely to enunciate and explain, preferring to devote himself chiefly to details of practice which are not always found in text-books.

Definition (i) **If three numbers, a, x, N, are connected by the equation—**

$$a^x = N,$$

then x is called "the logarithm of N to the base a".

This relationship is expressed symbolically thus:—

$$\log_a N = x.$$

Referring to the table on p. 102 we see that $8 = 2^3$, $128 = 2^7$, $8192 = 2^{13}$; hence 3, 7, 13 are the respective logarithms of 8, 128, 8192 *to the base* $\underline{2}$, *i.e.*, $\log_2 8 = 3$, $\log_2 128 = 7$, $\log_2 8192 = 13$.

The definition is sometimes given in the following form:—

(ii) **The logarithm of a number to a given base is the index of that power of the base which is equal to the number.**

Following the scheme described on p. 102, but *trebling* instead of *doubling*, we might construct a similar table showing logarithms *to the base* $\underline{3}$, and it is obvious also that similar tables could be constructed to any other base.

As a rule, practical calculators have little to do with logarithms to any other base than 10. For this number, *being the radix of the common system of notation*, allows of much greater simplicity and compactness in the construction of tables than any other.

Logarithms to the base 10 are called **Briggs's** or **Common** Logarithms.

91. One other system of Logarithms may sometimes require attention—that calculated to base 2·7182818284.. the number found, to any required degree of accuracy, by summing the infinite series—

$$1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \text{etc.}$$

and usually denoted by e.

Such logarithms are called **Napierian, Natural,** or **Hyperbolic** logarithms. On the few occasions on which a natural logarithm is wanted it may be found from the approximate equation—

$$\text{Nat. log.} = \frac{\text{Com. log.}}{\cdot 4342945} \quad \text{(i)}$$

$$\text{or Nat. log.} = \text{Com. log.} \times 2{\cdot}3025851 \quad \text{(ii)}$$

and a useful check on accuracy is provided by the rougher approximation—

$$\text{Nat. log.} = \left(\tfrac{99}{43} + \tfrac{1}{4000}\right) \text{ of Com. log. (iii)}$$

or by the still rougher one—

$$\text{Nat. log.} = \tfrac{99}{43} \text{ of Com. log. (iv).}$$

Thus, required the natural logarithm of 2, given that the common logarithm of 2, true to 7 figures, is ·3010300.

$$\begin{array}{lr}
 & \cdot 693147 \\
\text{By (i)}\ 4{,}3{,}4{,}2{,}9{,}4{,}5\ ) & 3010300 \\
 & 404533 \\
 & 13668 \\
 & 639 \\
 & 205 \\
 & 31
\end{array}
\qquad\qquad
\begin{array}{lr}
\text{By (ii)} & \underline{\cdot 30103\ \ } \\
 & \cdot 60206\ \ \\
 & 90309 \\
 & 602 \\
 & 150 \\
 & 24 \\
 & \underline{1} \\
 & \cdot 693146
\end{array}$$

By (iii)	30·103	By (iv)	·69307
	·30103		
43)	29·80197		
	·032		
	41		
	·69307		
	·00007		
	·69314		

Ex. (131) (i) Given that the common logarithm of 3 is ·4771213, find the natural logarithm by each of the above processes.

(ii) Given that the natural logarithm of 7 is 1·9459101, find the common logarithm.

92. If, in some exceptional piece of work, this transformation happened to be frequently required, it would help the computer to make a table of multiples of 4342945 for (i), or of 2·3025851 for (ii).

Multiples of ·434294 and of 2·302585 (1-100 times) are given in Bremiker's Tables, p. 186.

The hyperbolic logarithms of 1·01, 1·02, etc., to 10 are given in Hutton's Tables.

93. The principal theorems on which the practical rules for working with logarithms depend are as follows:—

I. **The logarithm of a product is equal to the sum of the logarithms of its factors.**

Thus $\log 28 = \log 4 + \log 7$;
and generally $\log (ab) = \log a + \log b$.

II. **The logarithm of a quotient is equal to the excess of the logarithm of the dividend over that of the divisor.**

Thus log $(7 \div 4) = \log 7 - \log 4$;
and generally $\log\left(\frac{a}{b}\right) = \log a - \log b$.

The rule may be more briefly given, thus:—

Log (fraction) = log (numerator) − log (denominator).

The logarithm of any power of a number is equal to the product of the logarithm of the number and the index of the power.

Thus $\log(3^4) = 4 \log 3$;
and generally $\log a^b = b \log a$.

N.B. The logarithms of roots of given numbers fall under this rule.

$$E.g.\quad \sqrt[5]{2} = 2^{\frac{1}{5}}.$$
$$\text{Hence } \log \sqrt[5]{2} = \tfrac{1}{5} \log 2$$
$$\log \sqrt[3]{(28)^7} = \tfrac{7}{3} \log 28$$

and generally—

$$\log \sqrt[b]{a} = \frac{1}{b} \log a$$

$$\log \sqrt[b]{a^c} = \frac{c}{b} \log a.$$

94. Any one of the above theorems is true for all systems of logarithms, *provided the same base be taken throughout.*

In what follows the base will be supposed 10 unless some other value be expressly stated.

Common logarithms have the following exceptional property:—

The logarithms of any two numbers expressed by the same set of digits in the same order differ by an integer only.

This follows easily from rule I. For if two numbers are denoted by the same set of digits in the same order (such,

for instance, as 17358, 173·58), one of them must be equal to the product of the other and *some integral power of* 10.

$$\text{Thus } 17358 = 10^2 \times 173{\cdot}58$$
$$\text{Hence } \log 17358 = \log (10)^2 + \log 173{\cdot}58$$
$$= 2 + \log 173{\cdot}58$$

If, therefore, from the tables or any other source, we learn that—

$$\log 17358 = 4{\cdot}2394997$$

we easily see that—

$$\log 173{\cdot}58 = 2{\cdot}2394997$$
$$\log 173580 = 5{\cdot}2394997$$
$$\log {\cdot}017358 = -2 + {\cdot}2394997$$

For convenience the last logarithm is written thus: $\bar{2}{\cdot}2394997$. In dealing with such expressions the student must be careful to remember that *the decimal part is positive.*

When a logarithm is not an exact integer, it is usual to express it in a form showing **the next integer below it** and **its excess above that integer**; the integer is then called the *characteristic*, and its excess above the characteristic the *mantissa* of the logarithm.

Note that the characteristic is sometimes *positive*, sometimes *negative*, whereas the mantissa is *always positive.*

Thus the logarithms of 17358, 173·58, 173580, ·017358 have 4, 2, 5 and -2 respectively for their *characteristics*, while all the same *mantissa*, viz., ·2394997.

As a rule the student will only find the *mantissa* of a logarithm supplied to him by Tables, and he will have to supply the *characteristic* himself. This he can easily do from an inspection of the given number whose logarithm is required. All he has to do is to **count how many places**

the first significant digit of the number is beyond the unit place, and write down the number thus found for the characteristic, making it positive or negative as the digit is to the left or right of the unit place.

Thus in the number 17358 the first significant digit, 1, is 4 *places beyond the unit place to the left*, hence the characteristic of its logarithm must be 4; while in the number ·017358 the first significant digit is 2 *places beyond the unit place to the right*, and hence the characteristic of its logarithm must be -2.

The rule may be more briefly stated thus: the characteristic of the logarithm of a number is identical with the "order" of its first significant digit.

95. Especial care must be taken both in *multiplying* and in *dividing* a logarithm which has a negative characteristic. Thus (i) if, in order to obtain log $(\cdot 017358)^5$, we multiply log ·017358 by 5 we obtain $\bar{9}\cdot 1974985$, the -9 being made up of $-10+1$. (ii) If, in order to obtain log $\sqrt[5]{\cdot 017358}$, we divide log ·017358 by 5 we obtain $\bar{1}\cdot 6478999$, *replacing mentally the* $\bar{2}$ *of the logarithm by* $-5+3$ before commencing the division.

Ex. (132) Write down the logs of ·17358, ·00017358 and 17358000.

Ex. (133) Find the logarithms of $(\cdot 017358)^2$, $(\cdot 0017358)^5$, $\sqrt[3]{\cdot 017358}$, $\sqrt[5]{\cdot 00017358}$.

Ex. (134) Write down the logarithms of 2000, ·002, ·2, 2^{10}, 30000, ·03, 3^{100}.

Ex. (135) How many figures are there (i) in the fiftieth power of 2, (ii) in the hundredth power of 3?

96. Extract from Chambers' (7-fig.) Logarithms of Numbers.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
2250	3521825 ·	2018	2211	2404	2597	2790	2983	3176	3369	3562	193
51	3755	3948	4141	4334	4527	4720	4912	5105	5298	5491	1·19
52	5684	5877	6070	6262	6455	6648	6841	7034	7226	7419	2·39
53	7612	7805	7997	8190	8383	8576	8768	8961	9154	9346	3·58
54	9539	9732	9924	$\overline{0117}$	$\overline{0310}$	$\overline{0502}$	$\overline{0695}$	$\overline{0888}$	$\overline{1080}$	$\overline{1273}$	4·77
55	3531465	1658	1851	2043	2236	2428	2621	2813	3006	3198	5·97
56	3391	3583	3776	3968	4161	4353	4546	4738	4931	5123	6·116
57	5316	5508	5700	5893	6085	6278	6470	6662	6855	7047	7·135
58	7239	7432	7624	7816	8009	8201	8393	8586	8778	8970	8·154
59	9162	9355	9547	9739	9931	$\overline{0123}$	$\overline{0316}$	$\overline{0508}$	$\overline{0700}$	$\overline{0892}$	9·174

Here the *numbers* are given to 5 and the *corresponding logarithms* to 7 significant figures, so that by a mere reading from the tables we can answer questions like the following :—

(i) What are the 7-fig. logarithms of 2253·7 and ·22546?

Answer. log 2253·7 = 3·3528961
log ·22546 = $\bar{1}$·3530695

the bar over 0695 in the tables denoting that a 1 has to be "carried" to the 352.

Similarly, log ·00022598 = $\bar{4}$·3540700.

Ex. (136) Find log 225·47 and log ·0022574.

(ii) Find to 5 significant figures the number whose 7-fig. logarithms are 4·3522468 and $\bar{3}$·3531259.

Answer. log 22503 = 4·3522468
log ·0022549 = $\bar{3}$·3531259.

Ex. (137) Find numbers whose logs are 3·3527805 and $\bar{4}$·3537816.

By means of a small calculation, in which use may be made of the small table at the side, we are enabled to answer questions requiring a greater degree of accuracy, though these are comparatively seldom proposed *except to candidates for competitive examinations.*

(i) Thus, find the logarithms 225·678 and ·02259623.

log 225·67	=	2·3534738
Diff. for 8	=	154
log 225·678	=	2·3534892
log ·022596	=	$\bar{2}$·3540316
Diff. for 2	=	39
Diff. for 3	=	6
log ·02259623	=	$\bar{2}$·3540361

(ii) Find to 7 significant figures the numbers whose logarithms are $\bar{4}\cdot3530720$ and $6\cdot3536349$.

$$
\begin{array}{rcr}
\log x & = & \bar{4}\cdot3530720 \\
\log \cdot00022546 & = & \underline{\bar{4}\cdot3530695} \\
 & & 25 \\
\text{Diff. for } 1 & = & \underline{19} \\
\text{Diff. for } 3 & = & 6
\end{array}
$$

$$x = \cdot0002254613.$$

$$
\begin{array}{rcr}
\log x & = & 6\cdot3536349 \\
\log 2257500 & = & \underline{6\cdot3536278} \\
 & & 71 \\
\text{Diff. for } 30 & = & \underline{58} \\
\text{Diff. for } 1 & = & 13
\end{array}
$$

$$x = 2257531.$$

For the *theory* of Proportional Parts the student is referred to Levett and Davison's *Plane Trigonometry* (pp. 380-388); we have here to do merely with the *practice*.

Ex. (138) Find log 22·5679 and log ·0002258734.

Ex. (139) Find numbers whose logarithms are $\bar{6}\cdot3537928$ and 4·3529840.

97. Extract from Bremiker's (6-fig.) Logarithms of Numbers (D. Nutt).

N	0	1	2	3	4	5	6	7	8	9	PP	
1250	·096910	6945	6979	7014	7049	7084	7118	7153	7188	7223		35
51	7257	7292	7327	7361	7396	7431	7466	7500	7535	7570	1	3 . 5
52	7604	7639	7674	7708	7743	7778	7812	7847	7882	7916	2	7 . 0
53	7951	7986	8020	8055	8090	8124	8159	8194	8228	8263	3	10 . 5
54	8298	8332	8367	8401	8436	8471	8505	8540	8575	8609	4	14 . 0
55	098644	8678	8713	8748	8782	8817	8851	8886	8920	8955	5	17 . 5
56	8990	9024	9059	9093	9128	9162	9197	9232	9266	9301	6	21 . 0
57	9335	9370	9404	9439	9473	9508	9543	9577	9612	9646	7	24 . 5
58	9681	9715	9750	9784	9819	9853	9888	9922	9957	9991	8	28 . 0
59	100026	0060	0095	0129	0164	0198	0233	0267	0302	0336	9	31 . 5

Here the *numbers* are given to 5 and the *corresponding logarithms* to 6 places. The differences are consequently much smaller than in the 7-figure table and can be added mentally. Thus we can not only write down directly from the table—

$$\log 125{\cdot}68 = 2{\cdot}099266$$
$$\log {\cdot}000012584 = \bar{5}{\cdot}099819$$

but, by means of the small "PP" table at the side and a slight mental calculation—

$$\log 12543{\cdot}7 = 4{\cdot}098426$$
$$\log 1{\cdot}25866 = {\cdot}099909.$$

Ex. (140) *Write down* log 125735 and log ·125268.

Ex. (141) *Write down* the numbers whose logs are $\bar{4}{\cdot}097965$ and $3{\cdot}100261$.

98. Extract from Lang's (5-fig.) Logarithms of Numbers. A small edition (Blackwood & Sons) containing the logarithms of the numbers from 1 to 9999. It is small enough to be carried in the waistcoat pocket.

	0	1	2	3	4	5	6	7	8	9
370	56820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	♦08	♦19	♦31	♦43
372	57054	066	078	089	101	113	124	136	148	159

Here the *numbers* are given to 4 and the *corresponding logarithms* to 6 places; the "black diamond" indicates 10, and replaces the "bar" of Chambers' Table. In extracting a logarithm put 0 for the diamond and "carry one" to the preceding figures; thus, log 371·6 = 2·57008. There is no "PP" table, but, as the differences are small, operations on them can be done mentally.

Ex. Find log 37·146.

Here the "difference for 10" (*i.e.*, between log 37·140 and log 37·150) is 12. Hence the difference for 6 is 7, and log 37·146 = 1·56991.

Ex. Find the number whose logarithm is ·57132.

Here log 3·726 = ·57124, and the difference 8 is easily seen to be the difference for 7.

Hence the required number is 3·7267.

Ex. (142) Find log 37·148, log 37004, log ·37239, log ·0037106.

Ex. (143) Construct a "PP" table for the above extract.

99. Extract from (4-fig.) Card of Logarithms and Antilogarithms (P. Young, 137 Gower Street).

This card (10″·5 × 8″·5) gives (*a*) on one side, the logarithms to 4 places of decimals of the numbers 10—999, with differences which make it easy to write down the logarithms of numbers of 4 digits; (*b*) on the other side, the antilogarithms to 4 figures of the mantissæ ·000—·999, with differences which make it easy to write down the numbers corresponding to logarithms of 4 decimal places.

(*a*) Logarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7

From the above we can write down not only log 55·2 = 1·7419, log ·0000557 = $\bar{5}$·7459, but also, by means of the table of differences, log 5503 = 3·7406, log ·5567 = $\bar{1}$·7456.

(*b*) Antilogarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7

From the above we can not only write down directly the numbers (318·4 and ·03214) corresponding respectively to

the logarithms 2·503, $\bar{2}$·507, but, by means of the table of differences, the numbers (3185 and ·3220) corresponding to the logarithms 3·5032 and $\bar{1}$·5078.

Ex. (144) Write down log 5·54, log 556, log ·00558, log 550·7 and log ·05574.

Ex. (145) Write down the antilogarithms of ·3184, 1·3199, $\bar{2}$·3214, 2·3180, $\bar{3}$·3200 and $\bar{4}$·3225.

100. The student is recommended to work out a few examples with all the different tables of logarithms available. As specimens a few calculations are added, in which all the four tables already described are used for the same piece of work.

Ex. (i) Find the product of 1·2599 and 178·87.

	Cs.	Br.		Sg.	Yg.
log 1·2599	·1003361	·100336	log 1·260	·10037	·1004
log 178·87	2·2525375	2·252538	log 178·9	2·25261	2·2526
	2·3528736	2·352874		2·35298	2·3530

If we look for the number most nearly corresponding to the logarithm of the product, in either Chambers or Bremiker, we find 225·36; if in either Sang or Young, we find 225·4.

Ex. (146) Find the product of 25·921 and 2·3961.

Ex. (ii) Find the quotient of ·40961 by 1030·2.

	Cs.	Br.		Sg.	Yg.
log ·40961	$\bar{1}$·6123706	$\bar{1}$·612371	log ·4096	$\bar{1}$·61236	$\bar{1}$·6123
log 1030·2	3·0129215	3·012922	log 1030	3·01284	3·0128
log quot.	$\bar{4}$·5994491	$\bar{4}$·599449		$\bar{4}$·59952	$\bar{4}$·5995

Quotient by Chambers or Bremiker ·00039760.
Quotient by Sang or Young . . ·0003977.

Ex. (147) Find the quotient of 9·691 by ·11157.

Ex. (iii) Find the fifth power of ·0533.

log ·0533	$\bar{2}$·7267272	$\bar{2}$·726727	$\bar{2}$·72673	$\bar{2}$·7267
5 log ·0533	$\bar{7}$·6336360	$\bar{7}$·633635	$\bar{7}$·63365	$\bar{7}$·6335

Fifth power by Chambers or Bremiker ·00000043017.
Fifth power by Sang . . . ·0000004302.
Fifth power by Young . . . ·0000004300.

Ex. (148) Find the fourth power of ·00468.

Ex. (iv) Find the cube root of 2.

	7-fig.	4-fig.
log 2	·3010300	·3010
$\frac{1}{3}$ log 2	·1003433	·1003
$\sqrt[3]{2}$	1·2599	1·260

Ex. (149) Find the fifth root of 3 by 7-fig. and by 4-fig. tables.

101. The attention of the student is particularly drawn to the power of calculation afforded by the (4-fig.) logarithm card. He should certainly obtain this, and practise its use, whatever other more extensive table he has.

102. The above examples have purposely been chosen of a simple character. As a matter of fact it would probably be as expeditious to work (i) and (ii) without logarithms as with them, if the methods described on pp. 65, 68 were used; while (iii) and (iv), though more quickly worked by means of logarithms, could be easily done without them. But if more than two factors are to be multiplied together, the advantages of logarithms become apparent, while if the index of a required power is a mixed number, we obtain readily by means of logarithms what it would be almost impossible, in many cases, to obtain without them.

103. In practice work is often facilitated by using what is called the **cologarithm** of a number, *i.e., the logarithm of*

its reciprocal, instead of using the logarithm itself. It is to be noticed that—

$$\text{since } \log\left(\frac{1}{a}\right) = -\log a$$
$$\therefore \ \operatorname{colog} a = -\log a$$
$$\therefore \ \operatorname{colog} a + \log a = 0$$

The cologarithm of a number is therefore written down *from left to right* by the same rule as the *arithmetical complement*, see p. 6.

$$\text{Thus since } \log 225{\cdot}47 = 2{\cdot}3530888$$
$$\operatorname{colog} 225{\cdot}47 = \bar{3}{\cdot}6469112$$
$$\text{Again, since } \log {\cdot}01251 = \bar{2}{\cdot}097257$$
$$\operatorname{colog} {\cdot}01251 = 1{\cdot}902743$$

The student is strongly recommended to use cologarithms as in the examples which follow. The advantages of their use will probably not be obvious at first, but will occur to him as he proceeds. He should also practise writing down the cologarithm of a number by inspection of the table of logarithms without writing down the logarithm first; it may assist him to draw his attention to the two following rules:—

(i) *If a number has any integral digits, the number of those integral digits, with a minus sign prefixed, gives the integral part of its cologarithm.*

Thus the integral part of colog 517·8 is -3.

(ii) *If a number has no integral digits, the integral part of its cologarithm is equal to the number of cyphers following the decimal point.*

Thus the integral part of colog ·0456 is 1; that of colog ·000456 is 3.

Ex. (150) From an inspection of the extracts on p. 118 write down the cologarithms of 2258·7, 2·2542, ·22581, ·00022506.

Ex. (151) From an inspection of pp. 114, 115 write down the cologs of 370400, ·00003716, 55700000, ·0552.

Ex. Evaluation of $\dfrac{10{\cdot}3993 \times {\cdot}033215 \times {\cdot}208341}{{\cdot}0033102 \times 1{\cdot}04348 \times 66{\cdot}317}$

$$
\begin{array}{rl}
\log {\cdot}0033102 = & \bar{3}{\cdot}5198542 \\
\log 1{\cdot}0434 = & \phantom{\bar{3}}{\cdot}0184508 \\
\text{diff. for } 8 = & \phantom{\bar{3}{\cdot}5198}334 \\
\log 66{\cdot}317 = & \underline{1{\cdot}8216249} \\
\log \text{denom.} = & \underline{\bar{1}{\cdot}3599633} \\
\text{colog denom.} = & \phantom{\bar{3}}{\cdot}6400367 \\
\log 10{\cdot}399 = & 1{\cdot}0169916 \\
\text{diff. for } 3 = & \phantom{\bar{3}{\cdot}5198}125 \\
\log {\cdot}033215 = & \bar{2}{\cdot}5213343 \\
\log {\cdot}208341 = & \underline{\bar{1}{\cdot}3187748} \\
\log \text{fraction} = & \bar{1}{\cdot}4971499 \\
\log {\cdot}31415 = & \bar{1}{\cdot}4971\underline{371} \\
\text{T. diff. } 138 & \phantom{\bar{1}{\cdot}49}\,)\,128 \\
 & \phantom{\bar{1}{\cdot}497}\underline{38} \\
 & \phantom{\bar{1}{\cdot}4971}93
\end{array}
$$

Fraction $= {\cdot}3141593$.

104. It is to be noticed that it is really unnecessary to find the *logarithm* of the denominator at all, as we may form the cologarithm figure by figure as we add up (see pp. 5, 6). The work might be checked by finding the logarithm of the numerator first, and then subtracting the sum of the logarithms of the factors of denominator from it, *in one operation*, as described on p. 5.

Ex. (152) Evaluate (i) $\dfrac{{\cdot}00333 \times 1{\cdot}13 \times 103{\cdot}993}{10{\cdot}6 \times {\cdot}0355 \times 33{\cdot}102}$;

(ii) $\dfrac{13{\cdot}957 \times 1{\cdot}7 \times 810 \times 23}{594{\cdot}76 \times 4 \times 19 \times 9{\cdot}8}$;

(iii) $\dfrac{{\cdot}065}{27{\cdot}7} \times \dfrac{37{\cdot}5}{{\cdot}0088} \times \dfrac{103{\cdot}3}{4{\cdot}402} \times \dfrac{{\cdot}2154}{9179}$;

(iv) $\dfrac{{\cdot}863}{15{\cdot}5} \times \dfrac{15{\cdot}20}{{\cdot}0273} \times \dfrac{{\cdot}016063}{28{\cdot}85} \times \dfrac{6{\cdot}57}{{\cdot}118}$.

Evaluation of $\dfrac{\sqrt[7]{(466871)^6} \times \sqrt[9]{(3576)^{16}}}{996003\ \sqrt{\cdot 0071}}$

$$
\begin{array}{rl}
\log \cdot 0071 = & \bar{3}\cdot 8512583 \\
\tfrac{1}{2}\text{-}\log \cdot 0071 = & \bar{2}\cdot 9256292 \\
\log 99600 = & 5\cdot 9982593 \\
\text{Diff. for } 3 = & 13 \\
\log \text{denom.} = & 4\cdot 9238898 \\
\\
\log 3576 = & 3\cdot 5533975 \\
2 \log 3576 = & 7\cdot 1067950 \\
 & \cdot 7896439 \\
\tfrac{16}{9}\text{-}\log 3576 = & 6\cdot 3171511 \\
\\
\log 466871 = & 5\cdot 6691969 \\
 & \cdot 8098853 \\
 & 4\cdot 8593116 \\
 & 6\cdot 3171511 \\
\text{colog denom.} = & \bar{5}\cdot 0761102 \\
\log \text{fraction} = & 6\cdot 2525729 \\
\log 1788800 = & 6\cdot 2525618 \\
\text{T. Diff. } 24_{\wedge}3 &)\ 111 \\
 & 138 \\
 & 46
\end{array}
$$

Fraction = 1788846.

In the work we have made use of the identities $\frac{6}{7} = 1 - \frac{1}{7}$; $\frac{16}{9} = 2 - \frac{2}{9}$.

Ex. (153) Evaluate (i) $\dfrac{\sqrt{52\cdot 3} \times \sqrt[4]{\cdot 0037} \times (2\cdot 361)^5}{\sqrt[8]{(271\cdot 4)^7} \times \sqrt{(345\cdot 16)^8}}$;

(ii) $\dfrac{321576}{942184} \times \sqrt{\dfrac{97}{28420}} \times \sqrt[3]{\dfrac{184}{5391}} \times \sqrt[3]{\dfrac{100}{(293)^2}}$.

Ex. Solve the equation $2^x = 384{\cdot}5$.

$$x \log 2 = \log 384{\cdot}5$$

$$x = \frac{\log 384{\cdot}5}{\log 2}$$

$$= \frac{2{\cdot}584896}{{\cdot}301030}$$

$$= 8{\cdot}58684.$$

$$\begin{array}{r} 8{\cdot}58684 \\ 301030\,)\,2584896 \\ 176656 \\ 26141 \\ 2059 \\ 256 \\ 12 \end{array}$$

Ex. (154) Solve the equations—

(i) $2^x = 96{\cdot}125$; (ii) $3^x = 384{\cdot}5$; (iii) $10^x = 4718$; (iv) $12^x = 573$.

Ex. Solve the equation—

$$(\tfrac{17}{13})^{8-x}\,(1{\cdot}42)^{x+3} = 19$$

$$(8 - x)\log\tfrac{17}{13} + (x + 3)\log 1{\cdot}42 = \log 19$$

$$x\,(\log 1{\cdot}42 - \log\tfrac{17}{13}) = \log 19 - 8\log\tfrac{17}{13} - 3\log 1{\cdot}42.$$

$$\begin{array}{rl} \log 17 = & 1{\cdot}230449 \\ \log 13 = & 1{\cdot}113943 \\ \hline & {\cdot}116506 \\ \log 1{\cdot}42 = & {\cdot}152288 \\ \hline x\,({\cdot}035782) = & \end{array} \qquad \begin{array}{rr} \log 19 = & 1{\cdot}278754 \\ 8\log\tfrac{17}{13} = & {\cdot}932048 \\ 3\log 1{\cdot}42 = & {\cdot}456864 \\ \hline & \bar{1}{\cdot}889842 \\ \hline & -\ {\cdot}110158 \end{array}$$

$$x = -\frac{110158}{35782} \qquad \begin{array}{r} 3{\cdot}0786 \\ 35782\,)\,110158 \\ 2812 \\ 307 \\ 21 \end{array}$$

$$= -\ 3{\cdot}0786$$

Ex. (155) Solve the equations—

(i) $2^{6+x} \times 3^{7-x} = 15$; (ii) $(2{\cdot}97)^{x+5} \times \sqrt[3]{81} = (17)^{5-x}$; (iii) $17^{x+3} \times 13^{x+2} = (221)^4$.

105. By means of the method for the solution of certain equations of the form: $ax = b + q$, on p. 100, we may sometimes solve equations involving the first power of x as well as terms of the form a^x.

Ex. Solve the equation—

$$7x + 2^x = 71.$$

Here, if we write $7x = 71 - 2^x$ and try to use the method employed, we find that 2^x changes too rapidly with x for the method to be available. But if we transpose and take logarithms we get—

$$x = \frac{\log (71 - 7x)}{\log 2}.$$

Seeing that $x = 5$ is a near approximation, we substitute 5 on the right-hand side and get—

$$x = \frac{\log 36}{\log 2} = 5{\cdot}17.$$

Substituting 5·17 we get—

$$x = \frac{\log 34{\cdot}81}{\log 2} = 5{\cdot}12.$$

Substituting 5·12 we get—

$$x = \frac{\log 35{\cdot}16}{\log 2} = 5{\cdot}13,$$

which from its close agreement with the preceding we expect to be a very good approximation.

Substituting, as a check, 5·13 we get—

$$x = \frac{\log 35{\cdot}09}{\log 2},$$

which again gives $x = 5{\cdot}13$.

Hence we conclude that this value is true to 3 significant figures.

The work may be arranged thus :—

36	34·81	35·16	35·09
5·17	5·12	5·13	5·13
3010) 15563	3010) 15417	3010) 15460	3010) 15452
513	367	410	402
212	66	109	101

LOGARITHMS OF TRIGONOMETRICAL RATIOS.

106. We have now to introduce the student to various tables of the logarithms of the trigonometrical ratios or

functions (sine, cosine, tangent, etc.) of angles between 0° and 90°. That the logarithms of these ratios should be frequently wanted, and that consequently comprehensive tables of them have been calculated and published, he will naturally expect; but he may find some trouble at first in using them, from the fact that it is traditional to confront him with two quite unnecessary difficulties.

(i) He will continually want in his work the value of some such function as *the logarithm of the sine of* 29° 4′ (written, log sin 29° 4′), but he will not, as he might naturally expect, find this supplied in his table; what he finds is the *real value of this function increased by* 10. Thus opposite the 4′, in the column headed **sine** of the accompanying extract, he will find 9·6864816, which is 10 + *log sin* 29° 4′.

(ii) A special name, *logarithmic sine*, has been invented for this unnecessary function, and a special symbol "L sin" to denote it. He is strongly recommended to keep his own work clear of all these absurdities, to extract from the tables only *the logarithms he wants* and *to reject the superfluous* 10.*

From the identities—

$$\cos A = \sin (90° - A)$$
$$\cot A = \tan (90° - A)$$
$$\operatorname{cosec} A = \sec (90° - A)$$

it is plain that every numerical entry has two different names. Thus 9·7446453 is at once "L tan 29° 3′" and "L cot 60° 57′". This is the last occasion on which we shall use "big L". In future when we extract from the tables we shall reject the 10 and write as follows:—

$$\log \tan 29° \, 3' = \bar{1}\cdot 7446453 = \log \cot 60° \, 57'.$$

* De Morgan expressed himself strongly in favour of the views here advocated. As to the entries in the tables he says: "For myself I feel assured that the student should be taught *real logarithms* and left to find his own way to the other practice, which I much doubt his doing".

As to the name "logarithmic sine" he says: "The phrase is as incorrect as *royal country* would be for *king of the country*, or *constabulary parish* for *constable of the parish*".

107. Extract from Chambers' Table of Logarithms of Trigonometrical Ratios.

29° '	Sine	Diff	Cosec	Tan	Diff	Cot	Sec	Diff	Cos	'
0	9·6855712		10·3144288	9·7437520		10·2562480	10·0581807		9·9418193	60
1	9·6857991	2279	10·3142009	9·7440499	2979	10·2559501	10·0582508	701	9·9417492	59
2	9·6860267	2276	10·3139733	9·7443476	2977	10·2556524	10·0583209	701	9·9416791	58
3	9·6862542	2275	10·3137458	9·7446453	2977	10·2553547	10·0583910	701	9·9416090	57
4	9·6864816	2274	10·3135184	9·7449428	2975	10·2550572	10·0584612	702	9·9415388	56
5	9·6867088	2272	10·3132912	9·7452403	2975	10·2547597	10·0585315	703	9·9414685	55
6	9·6869359	2271	10·3130641	9·7455376	2973	10·2544624	10·0586018	703	9·9413982	54
		2269			2973			703		
'	Cos	Diff	Sec	Cot	Diff	Tan	Cosec	Diff	Sine	' 60°

In these Tables the time-honoured but absurd rule of increasing every logarithm by 10 has been followed. In using them the student should reject the 10, writing down—

$$\begin{aligned} \log\sin 29^\circ\,2' &= \bar{1}\cdot6860267 \\ \log\tan 29^\circ\,6' &= \bar{1}\cdot7455376 \\ \log\cos 29^\circ\,3' &= \bar{1}\cdot9416090 \\ \log\tan 60^\circ\,59' &= \cdot2559501 \\ \log\sin 60^\circ\,55' &= \bar{1}\cdot9414685 \end{aligned}$$

The columns headed "cosec" and "sec" might be omitted without much loss, since—

$$\begin{aligned} \log\operatorname{cosec} A &= \operatorname{colog}\sin A \\ \text{and } \log\sec A &= \operatorname{colog}\cos A. \end{aligned}$$

It should be noted that by means of the numbers headed "Diff" we can obtain the logarithms of the ratios of angles which are not given directly by the table.

Ex. To find (i) log sin 29° 4′ 35″; (ii) log cos 29° 4′ 36″.

$$\begin{aligned} \text{(i) } \log\sin 29^\circ\,4' &= \bar{1}\cdot6864816 \\ \text{Diff for } 35'' &= \underline{\phantom{\bar{1}\cdot68}1325} \\ \log\sin 29^\circ\,4'\,35'' &= \bar{1}\cdot6866141 \end{aligned} \qquad \begin{array}{r|r} 60'' & 2272 \\ \hline 30'' & 1136 \\ 5'' & 189 \\ \hline & 1325 \end{array}$$

$$\begin{aligned} \text{(ii) } \log\cos 29^\circ\,4' &= \bar{1}\cdot9415388 \\ \text{Diff for } 36'' &= \underline{\phantom{\bar{1}\cdot941}422} \\ \log\cos 29^\circ\,4'\,36'' &= \bar{1}\cdot9414966 \end{aligned} \qquad \begin{array}{r|r} 60'' & 703 \\ \hline 30'' & 352 \\ 6'' & 70 \\ \hline & 422 \end{array}$$

Here in (i), where we had to do with a *sine*, we *added* 35-sixtieths of the difference (2272) for 60″; in (ii), where

we had to do with a *cosine*, we *subtracted* 36-sixtieths of the difference (703) for 60″.

In similar cases we *add* for *sin*, *tan*, *sec*,

but we *subtract* for *cosin*, *cotan*, *cosec*.

It is often convenient to work the differences *by Practice* as in the above examples (see p. 24).

Ex. (156) Find log sin 29° 4′ 36″, log cos 29° 4′ 35″, log tan 29° 5′ 24″, log sec 29° 6′ 48″, log cosec 29° 0′ 55″.

The same difference columns will also enable us to perform the *inverse* process.

Ex. To find A (i) when log tan A = ·2546000;

(ii) when log cot A = $\bar{1}$·7451329

(i)		(ii)	
log tan A =	·2546000	log cot A =	$\bar{1}$·7451329
log tan 60° 54′ =	·2544624	log cot 60° 56′ =	$\bar{1}$·7449428
	1376		1901
	60	29‸7‸5)	114060
T. Diff 29‸7‸3	82560		2481
	2310		101
	229		38″·3
	27″·8		
A = 60° 54′ 27″·8.		A = 60° 55′ 21″·7	

Here in (i) we have added $\frac{1376}{2973}$ of 60″ to 60° 54′;

in (ii) we have subtracted $\frac{1901}{2975}$ of 60″ from 60° 56′, the angle taken from the table being that in each case whose tabulated function came next in value below the one given.

The calculation of these differences would often be much facilitated if the tabular differences were given for 100″ as in Maskelyne's, for 10″ as in Bremiker's, or for 1″ as in Vega's Tables, instead of being given for 60″ as in the above.

Ex. (157) Find A when (i) log sin A = $\bar{1}$·6863542; (ii) log tan A = $\bar{1}$·7447000; (iii) log cos A = $\bar{1}$·9417130; (iv) log sec A = ·0583759.

108. Extract from Bremiker's Table of Logarithms of the Trigonometrical Functions to every tenth second.

37°

′	″	Sin	d	Tan	d.c.	Cotg	Cos	d	″	′
50	0	9·787720		9·890204		0·109796	9·897516		0	10
			27		43			16		
			27		43			16		
59	0	9·789180		9·892549		0·107450	9·896631		0	1
			27		44			17		
	10	9·789207		9·892593		0·107407	9·896614		50	
			27		43			16		
	20	9·789234		9·892636		0·107364	9·896598		40	
			27		44			17		
	30	9·789261		9·892680		0·107320	9·896581		30	
			27		43			16		
	40	9·789288		9·892723		0·107277	9·896565		20	
			27		43			16		
	50	9·789315		9·892766		0·107234	9·896549		10	
			27		44			17		
60	0	9·789342		9·892810		0·107190	9·896532		0	0
′	″	Cos	d	Cotg	d.c.	Tan	Sin	d	″	′

52°

	17
1	1·7
2	3·4
3	5·1
4	6·8
5	8·5
6	10·2
7	11·9
8	13·6
9	15·3

From this we can extract at once log sin 37° 59′ 10″, log tan 37° 59′ 20″, log cot 37° 59′ 30″, log sin 52° 0′ 50″,

etc., and with a slight mental calculation, aided by the small subsidiary table on the right-hand side—

$$\log \cos 37^\circ\ 59'\ 27'' = \bar{1}{\cdot}896586$$
$$\log \tan 52^\circ\ 0'\ 43'' = {\cdot}107377$$

Ex. (158) *Write down* (i) log sin 37° 59′ 33″; (ii) log tan 37° 59′ 55″; (iii) log cos 52° 0′ 25″; (iv) log cot 37° 59′ 8″.

Ex. (159) Write down the value of A, given (i) log sin A = $\bar{1}{\cdot}789225$; (ii) log cos A = $\bar{1}{\cdot}896600$; (iii) log tan A = ·107255; (iv) log cot A = $\bar{1}{\cdot}892705$.

109. Before proceeding to the application of Logarithms to such practical problems as the determination of Heights and Distances, or the solution of Triangles by Trigonometry, the student is advised to exercise himself in the use of the Tables by means of easily-verified examples afforded by various Trigonometrical identities and approximations. The following are well-known identities :—

$$\sin \theta = \tan \theta \cos \theta$$
$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}.$$

Hence $\log \sin \theta = \log \tan \theta + \log \cos \theta$

$$\log \sin \theta = \log 2 + \log \sin \frac{\theta}{2} + \log \cos \frac{\theta}{2}.$$

Thus :—

$$\begin{aligned} \log \tan 37^\circ\ 59'\ 10'' &= \bar{1}{\cdot}892593 \\ \log \cos 37^\circ\ 59'\ 10'' &= \bar{1}{\cdot}896614 \\ \hline \text{Hence } \log \sin 37^\circ\ 59'\ 10'' &= \bar{1}{\cdot}789207 \end{aligned}$$

$$\begin{aligned} \log \sin 29^\circ\ 2' &= \bar{1}{\cdot}6860267 \\ \log \cos 29^\circ\ 2' &= \bar{1}{\cdot}9416791 \\ \log 2 &= {\cdot}3010300 \\ \hline \text{Hence } \log \sin 58^\circ\ 4' &= \bar{1}{\cdot}9287358 \end{aligned}$$

Ex. (160) Verify
(i) log sin 64° 16′ 40″ = log tan 64° 16′ 40″ + log cos 64° 16′ 40″
(ii) log sin 36° 42′ = log sin 18° 21′ + log cos 18° 21′ + log 2.

For angles less than 5° the following approximations hold to 5 figures of the logarithms:—

$$\cos\theta = \left(\cos\frac{\theta}{2}\right)^4; \quad \frac{\sin\theta}{\theta} = \sqrt[3]{\cos\theta}.$$

Hence $\log\cos\theta = 4\log\cos\frac{\theta}{2}$,

and $\log\sin\theta = \log\theta + \frac{1}{3}\log\cos\theta$.

Since θ is here the circular measure of the angle, we have, if n be the number of *degrees* in it, $\sin\theta = \frac{n\pi}{180}\sqrt[3]{\cos\theta}$,

and hence $\log\sin\theta = \log n + \log\frac{\pi}{180} + \frac{1}{3}\log\cos\theta$.

If n be the number of minutes in the angle, we have similarly $\log\sin\theta = \log n + \log\frac{\pi}{10800} + \frac{1}{3}\log\cos\theta$.

Thus—

$$\begin{aligned} \log\cos 4^\circ &= \bar{1}\cdot998941 \\ \tfrac{1}{4}\log\cos 4^\circ &= \bar{1}\cdot999735 \\ &= \log\cos 2^\circ \end{aligned}$$

$$\begin{aligned} \log\cos 5^\circ &= \bar{1}\cdot998344 \\ \tfrac{1}{3}\log\cos 5^\circ &= \bar{1}\cdot999448 \\ \log 5 &= \cdot698970 \\ \log\pi &= \cdot497150 \\ \text{colog } 180 &= \bar{3}\cdot744727 \\ &= \bar{2}\cdot940295 \\ \log\sin 5^\circ \quad & \bar{2}\cdot940296 \end{aligned}$$

Ex. (161) Verify (i) that $\log\cos 1^\circ = \frac{1}{4}\log\cos 2^\circ$; (ii) $\log\sin 4^\circ = \log 4 + \log\pi - \log 180 + \frac{1}{3}\log\cos 4^\circ$.

The following more extended formula affords practice with large angles as well as small:—

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cos\frac{\theta}{16}\cdots\sqrt[3]{\text{last cosine taken}}$$

the product being continued on the right-hand side until we reach an angle less than 5°.

Hence $\log \sin \theta = \log n + \log \frac{\pi}{180} + \log \cos \frac{\theta}{2}$

$+ \log \cos \frac{\theta}{4} \ . \ . \ . \ + \frac{1}{3} \log$ last cosine.

Ex. (162) Verify that—

$$\log \sin 80^\circ = \log 80 + \log \frac{\pi}{180} + \log \cos 40^\circ + \log \cos 20^\circ + \log \cos 10^\circ + \log \cos 5^\circ + \log \cos 2^\circ 30' + \frac{1}{3} \log \cos 2^\circ 30'.$$

Again putting $\theta = 30^\circ$ we have since $\sin 30^\circ = \frac{1}{2}$

$$\frac{3}{\pi} = \cos 15^\circ \cos 7^\circ 30' \cos 3^\circ 45' \sqrt[3]{\cos 3^\circ 45'}$$

Ex. (163) Verify that $\log 3 = \log \pi + \log \cos 15^\circ + \log \cos 7^\circ 30' + \log \cos 3^\circ 45' + \frac{1}{3} \log \cos 3^\circ 45'$.

Heights and Distances.

110. We now proceed to the evaluation of various Trigonometrical formulæ connected with the computation of heights and distances. For an *investigation* of these formulæ and an account of their application to various practical problems, works on Trigonometry must be consulted.

In the evaluation of the simplest of such formulæ (those of the form $a \tan \theta$, $\frac{a}{\tan \theta}$, $a \sin \theta$, $\frac{a}{\cos \theta}$, etc.) the use of logarithms has little or no advantage over ordinary multiplication and division if performed with the proper contractions and abbreviations.

Ex. To find $\frac{a}{\tan \theta}$ when $a = 967$, $\theta = 35^\circ 17'$,

(i) by division, (ii) by logarithms.

(i) $\tan 35^\circ 17' = \cdot 7076$

$a \cot \theta = 1366 \cdot 6$

$$\begin{array}{r} 7076 \,)\, 9670000 \\ 25940 \\ 4712 \\ 466 \\ 42 \end{array}$$

(*one* reference to tables).

(ii) $\log 967 = 2 \cdot 985426$

colog $\tan 35^\circ 17' = \cdot 150210$

$\log a \cot \theta = 3 \cdot 135636$

$a \cot \theta = 1366 \cdot 6$

(*three* references to tables).

The student cannot judge by the mere look of the two calculations which is really the more expeditious; he must go through each one by himself.

Ex. (164) Given $a = 258 \cdot 37$, $\theta = 54^\circ\ 13'\ 20''$, evaluate each of the following, (i) by ordinary division or multiplication, (ii) by logarithms :—

$\dfrac{a}{\sin \theta}$, $a \sin \theta$, $a \tan \theta$, $a \sec \theta$, $a \cot \theta$.

111. When, however, the expression to be evaluated has several factors the use of logarithms is very advantageous. The following is a common type of such expressions, and is taken with some other examples from the chapter on *Heights and Distances* in Hirsch's Geometry.

Ex. Evaluate $\dfrac{a \sin \alpha \sin \beta}{\sin (\beta - \alpha)}$ when $a = 967$, $\alpha = 7^\circ\ 5'\ 13''$, $\beta = 16^\circ\ 43'\ 5''$.

$$\begin{array}{rl} & 16^\circ\ 43'\ \ 5'' \\ & \underline{\ \ 7^\circ\ \ \ 5'\ 13''} \\ \beta - \alpha = & 9^\circ\ 37'\ 52'' \\ \log \sin 9^\circ\ 37'\ 52'' = & \underline{\bar{1} \cdot 223507} \\ \text{colog} = & \ \ \cdot 776493 \\ \log \sin 7^\circ\ 5'\ 13'' = & \bar{1} \cdot 091229 \\ \log \sin 16^\circ\ 43'\ 5'' = & \bar{1} \cdot 458883 \\ \log 967 = & \underline{2 \cdot 985426} \\ \log \text{fraction} = & 2 \cdot 312031 \\ \text{fraction} = & 205 \cdot 13. \end{array}$$

Ex. (165) Evaluate $\dfrac{a \sin \alpha \cos \beta}{\sin (\beta - \alpha)}$ the letters having the same values as in the above.

Ex. (166) Evaluate (i) $\dfrac{a \sin (\alpha + \beta)}{\cos \alpha}$

when $a = 1352{\cdot}7$; $\alpha = 19^\circ\ 7'$; $\beta = 25^\circ\ 13'$;

(ii) $\dfrac{a \sin \beta \tan \gamma}{\sin (\alpha + \beta)}$

when $a = 857$; $\alpha = 79^\circ\ 45'$; $\beta = 61^\circ\ 4'$; $\gamma = 14^\circ\ 19'\ 27''$;

(iii) $\dfrac{357{\cdot}3 \cos 3^\circ\ 48'\ 10'' \sin 79^\circ\ 13'\ 12'' \sin 7^\circ\ 12'\ 20''}{\sin 164^\circ\ 50'\ 26'' \cos 13^\circ\ 5'\ 49'' \cos 20^\circ\ 18'\ 9''}$;

(iv) $\dfrac{357{\cdot}3 \cos 3^\circ\ 48'\ 10'' \sin 79^\circ\ 13'\ 12'' \tan 20^\circ\ 18'\ 9''}{\sin 164^\circ\ 50'\ 26''}$.

Solution of Triangles.

112. We now proceed to the consideration of formulæ for the solution of triangles in general, first giving a list of those in common use which are adapted for logarithmic computation, and then working out a set of examples illustrating their use.

I. (α) Given two angles and a side opposite to one of them.
(β) Given two sides and the angle opposite to one of them.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2\,R.$$

II. Given two sides and the included angle.

$$\text{(i)}\quad \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)} = \frac{b - c}{b + c}.$$

$$\text{(ii)}\quad a = (b - c) \sec \phi \text{ when } \tan^2 \phi = \frac{4bc}{(b - c)^2} \sin^2 \frac{A}{2}.$$

$$\text{(iii)}\quad \tan \frac{A - B}{2} = \tan (45^\circ - \phi) \tan \frac{A + B}{2} \text{ when } \tan \phi = \frac{b}{a}$$

useful when the *logarithms only* of a and b are known.

III. Given the three sides—

$$\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc};$$

$$\cos^2\frac{A}{2}=\frac{s(s-a)}{bc};$$

$$\tan^2\frac{A}{2}=\frac{(s-b)(s-c)}{s(s-a)}.$$

To these may be added a few others which are occasionally useful, either for direct calculation or for checking results already obtained.

$$(b+c)\sin\frac{A}{2}=a\cos\frac{B-C}{2}$$

$$(b-c)\cos\frac{A}{2}=a\sin\frac{B-C}{2}$$

$$(b^2-c^2)\sin A=a^2\sin(B-C)$$

$$\begin{aligned}b^2-c^2&=2Ra\sin(B-C)\\&=4R^2\sin A\sin(B-C)\end{aligned}$$

113. I. (*a*) (i) Given $a=188{\cdot}5$, $A=41°\ 10'$, $B=54°\ 2'\ 22''$, find b.

$$\begin{aligned}\log\sin 41°\ 10'&=\bar{1}{\cdot}818392\\ \text{colog}&=\ {\cdot}181608\\ \log 188{\cdot}5&=2{\cdot}275311\\ \log\sin 54\ \ 2'\ 22''&=\bar{1}{\cdot}908175\\ \log b&=2{\cdot}365094\\ b&=231{\cdot}79.\end{aligned}$$

Ex. (167) Given—

(i) $a=396{\cdot}83$, $A=35°\ 13'$, $B=58°\ 4'\ 35''$; find b.

(ii) $a=4{\cdot}1738$, $A=19°\ 54'$, $C=37°\ 23'\ 40''$; find c.

(iii) $a=54391$, $B=38°\ 19'\ 23''$, $C=41°\ 18'$; find b.

I. (*a*) (ii) Given $c=79{\cdot}063$, $A=41°\ 13'\ 2''$, $B=67°\ 27'\ 53''$, solve the triangle—

$$C = 71^\circ\ 19'\ 5''.$$

$$2R = \frac{c}{\sin C};\ a = 2R \sin A;\ b = 2R \sin B.$$

$$\begin{aligned}
\log \sin 71^\circ\ 19'\ 5'' &= \bar{1}\cdot976493 \\
\text{colog} &= \cdot023507 \\
\log 79\cdot063 &= 1\cdot897973 \\
\log 2R &= 1\cdot921480 \\
\log \sin 41^\circ\ 13'\ 2'' &= \bar{1}\cdot818830 \\
\log a &= 1\cdot740310 \\
a &= 54\cdot9934. \\
\log 2R &= 1\cdot921480 \\
\log \sin 67^\circ\ 27'\ 53'' &= \bar{1}\cdot965505 \\
\log b &= 1\cdot886985 \\
b &= 77\cdot0876
\end{aligned}$$

Ex. (168) Solve the triangle completely from each of the following sets of data :—

(i) $c = 452\cdot37$, $A = 19^\circ\ 15'\ 38''$, $B = 67^\circ\ 27'\ 53''$.
(ii) $a = 9\cdot6421$, $A = 54^\circ\ 17'\ 19''$, $B = 71^\circ\ 11'\ 40''$.
(iii) $b = 75283$, $B = 23^\circ\ 54'\ 7''$, $C = 59^\circ\ 13'\ 50''$.
(iv) $a = 7000$, $B = 50^\circ$, $C = 80^\circ$.

I. (β) (i) Given $A = 43^\circ\ 28'$, $a = 1214$, $c = 1392$ to find C.

$$\sin C = \frac{c \sin A}{a}.$$

$$\begin{aligned}
\log 1214 &= 3\cdot084219 \\
\text{colog} &= \bar{4}\cdot915781 \\
\log \sin 43^\circ\ 28' &= \bar{1}\cdot837546 \\
\log 1392 &= 3\cdot143639 \\
\log \sin C &= \bar{1}\cdot896966 \\
C &= 52^\circ\ 4'\ 24'', \\
\text{or } C &= 127^\circ\ 55'\ 36''
\end{aligned}$$

Ex. (169) Given—

(i) $A = 54° 39'$, $a = 2316$, $c = 3529$, find C.

(ii) $A = 37° 48' 10''$, $a = 3578$, $b = 4138$, find B.

(iii) $B = 16° 13'$, $a = 360{\cdot}3$, $b = 291{\cdot}7$, find A.

(iv) $C = 19° 11' 23''$, $b = 27{\cdot}381$, $c = 15{\cdot}296$, find B.

I. (β) (ii) Given $b = 77{\cdot}041$, $c = 55$, $C = 41° 13' 22''$, solve the triangle—

$$2\,R = \frac{c}{\sin C}, \ \sin B = \frac{b}{2R} \ a = 2R \sin A.$$

$$\log \sin 41° 13' 22'' = \bar{1}{\cdot}818878$$

$$\text{colog} = \cdot 181122$$
$$\log 55 = 1{\cdot}740363$$

$$\log 2R = 1{\cdot}921485$$

$$\text{colog } 2R = \bar{2}{\cdot}078515$$
$$\log 77{\cdot}041 = 1{\cdot}886722$$

$$\log \sin B = \bar{1}{\cdot}965237$$

(1) $B = 67° 22' 48,$

or (2) $B = 112° 37' 12''$.

(1) $A = 71° 23' 50''$.

$$\log 2R = 1{\cdot}921485$$
$$\log \sin 71° 23' 50'' = \bar{1}{\cdot}976695$$

$$\log a = 1{\cdot}898180$$

$a = 79{\cdot}1001.$

(2) $A = 26° 9' 26''$.

$$\log 2R = 1{\cdot}921485$$
$$\log \sin 26° 9' 26'' = \bar{1}{\cdot}644277$$

$$\log a = 1{\cdot}565762$$

$a = 36{\cdot}793.$

Ex. (170) Solve the triangle completely from each of the following sets of data :—

(i) $a = 384$, $b = 760$, $A = 27^\circ\ 4'\ 32''$.
(ii) $b = 49{\cdot}7$, $c = 53{\cdot}4$, $B = 19^\circ\ 53'\ 11''$.
(iii) $a = 5{\cdot}3917$, $c = 7{\cdot}9482$, $A = 31^\circ\ 19'\ 23''$.

114. II. (i) Given $b = 3428{\cdot}43$, $c = 3277{\cdot}63$, $A = 68^\circ\ 2'\ 24''$ to find B and C.

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\tan\frac{B+C}{2}.$$

$$\begin{aligned}
b &= 3428{\cdot}43 \\
c &= 3277{\cdot}63 \\
\hline
b + c &= 6706{\cdot}06 \\
b - c &= 150{\cdot}80 \\
B + C &= 111^\circ\ 57'\ 36'' \\
\frac{B+C}{2} &= 55^\circ\ 58'\ 48''
\end{aligned}$$

$$\begin{aligned}
\log 6706{\cdot}06 &= 3{\cdot}826468 \\
\text{colog} &= \bar{4}{\cdot}173532 \\
\log 150{\cdot}8 &= 2{\cdot}178401 \\
\log\tan 55^\circ\ 58'\ 48'' &= {\cdot}170686 \\
\hline
\log\tan\frac{B-C}{2} &= \bar{2}{\cdot}522619 \\
\frac{B-C}{2} &= 1^\circ\ 54'\ 29'' \\
\frac{B+C}{2} &= 55^\circ\ 58'\ 48'' \\
B &= 57^\circ\ 53'\ 17'' \\
C &= 54^\circ\ 4'\ 19''
\end{aligned}$$

Ex. (171) Given—
(i) $a = 3754$, $b = 3277{\cdot}63$, $C = 54^\circ\ 4'\ 19''$, find A and B.

(ii) $b = 41{\cdot}592$, $c = 38{\cdot}673$, $A = 20°$, find B and C.
(iii) $a = 3{\cdot}2781$, $c = 4{\cdot}6827$, $B = 48°\ 1'\ 32''$, find A and C.
(iv) $b = 2c$, $A = 10°$, find B and C.

II. (ii) Given $a = 3754$, $b = 3277{\cdot}63$, $C = 54°\ 4'\ 19''$ to find c.

$$\tan\phi = \frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}.$$

$$c = (a-b)\sec\phi.$$

$$\begin{aligned}
a &= 3754 \\
b &= 3277{\cdot}63 \\
a - b &= 476{\cdot}37
\end{aligned}$$

$$\begin{aligned}
\log 3754 &= 3{\cdot}574494 \\
\log 3277{\cdot}63 &= \underline{3{\cdot}515560} \\
&\ \underline{7{\cdot}090054} \\
&\ 3{\cdot}545027 \\
\log 2 &= {\cdot}301030 \\
\text{colog } 476{\cdot}37 &= \bar{3}{\cdot}322056 \\
\log\sin 27°\ 2'\ 9''{\cdot}5 &= \underline{\bar{1}{\cdot}657582} \\
\log\tan\phi &= {\cdot}825695 \\
\log\cos\phi &= \underline{\bar{1}{\cdot}169511} \\
\log\sec\phi &= {\cdot}830489 \\
\log(a-b) &= \underline{2{\cdot}677944} \\
\log c &= 3{\cdot}508433 \\
c &= 3224{\cdot}3.
\end{aligned}$$

Ex. (172) Given $b = 3248{\cdot}43$, $c = 3277{\cdot}63$, $A = 68°\ 2'\ 24''$, find a.

Ex. (173) Find by the above method the third side in parts (i), (ii), (iii) of Ex. 170.

II. (iii) Given $\log a = 2{\cdot}693157$, $\log b = 2{\cdot}520827$, $C = 78^\circ\ 16'$ to find A and B.

$$\tan\frac{A-B}{2} = \tan(45^\circ - \phi)\tan\frac{A+B}{2}$$

$$\text{when } \tan\phi = \frac{b}{a}.$$

$$\begin{aligned}
\log b &= 2{\cdot}520827 \\
\log a &= 2{\cdot}693157 \\
\log\tan\phi &= \bar{1}{\cdot}827670 \\
\phi &= 33^\circ\ 55'\ 10'' \\
45^\circ - \phi &= 11^\circ\ 4'\ 50'' \\
\log\tan 11^\circ\ 4'\ 50'' &= \bar{1}{\cdot}291901 \\
\log\tan 50^\circ\ 52' &= \phantom{\bar{1}}{\cdot}089565 \\
\log\tan\frac{A-B}{2} &= \bar{1}{\cdot}381466 \\
\frac{A-B}{2} &= 13^\circ\ 32' \\
\frac{A+B}{2} &= 50^\circ\ 52' \\
A &= 64^\circ\ 24' \\
B &= 37^\circ\ 20'
\end{aligned}$$

Ex. (174) Given—

(i) $\log a = 3{\cdot}471283$, $\log b = 3{\cdot}648126$, $C = 34^\circ\ 8'$, find A and B.

(ii) $\log b = {\cdot}293748$, $\log c = {\cdot}471826$, $A = 62^\circ\ 9'\ 15''$, find B and C.

(iii) $\frac{a}{b} = 1{\cdot}62854$, $C = 41^\circ\ 19'\ 27''$, find A and B.

(iv) $\frac{a}{c} = 3{\cdot}56217$, $B = 53^\circ\ 14'\ 12''$, find A and C.

115. III. Given $a = 309{\cdot}86$, $b = 154{\cdot}33$, $c = 365$ to solve the triangle.

$$\text{Here } \tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)},\quad \tan^2\frac{B}{2} = \frac{(s-c)(s-a)}{s(s-b)}.$$

$$\begin{array}{rlrlrl}
a = & 309{\cdot}86 & \log 414{\cdot}595 = & 2{\cdot}617624 & \log 414{\cdot}595 = & 2{\cdot}617624 \\
b = & 154{\cdot}33 & \log 104{\cdot}735 = & \underline{2{\cdot}020092} & \log 260{\cdot}265 = & \underline{2{\cdot}415416} \\
c = & \underline{365} & \log s(s-a) = & \underline{4{\cdot}637716} & \log s(s-b) = & \underline{5{\cdot}033040} \\
2s = & \underline{829{\cdot}19} & \text{colog} = & \bar{5}{\cdot}362284 & \text{colog} = & \bar{6}{\cdot}966960 \\
s = & 414{\cdot}595 & \log 260{\cdot}265 = & 2{\cdot}415416 & \log 49{\cdot}595 = & 1{\cdot}695438 \\
s - a = & 104{\cdot}735 & \log 49{\cdot}595 = & \underline{1{\cdot}695438} & \log 104{\cdot}735 = & \underline{2{\cdot}020092} \\
s - b = & 260{\cdot}265 & & & & \\
s - c = & 49{\cdot}595 & 2\log\tan\frac{A}{2} = & \underline{\bar{1}{\cdot}473138} & 2\log\tan\frac{B}{2} = & \underline{\bar{2}{\cdot}682490} \\
& & \log\tan\frac{A}{2} = & \bar{1}{\cdot}736569 & \log\tan\frac{B}{2} = & \bar{1}{\cdot}341245 \\
& & \frac{A}{2} = & 28^\circ\ 36' & \frac{B}{2} = & 12^\circ\ 22'\ 29''{\cdot}5 \\
& & A = & 57^\circ\ 12' & B = & 24^\circ\ 44'\ 59'' \\
& & & & C = & 98^\circ\ 3'\ 1''
\end{array}$$

Ex. (175) Solve the triangle completely from each of the following sets of data :—

(i) $a = 297{\cdot}35$, $b = 346{\cdot}28$, $c = 579{\cdot}81$.

(ii) $a = 4$, $b = 5$, $c = 6$.

(iii) $a = 357{\cdot}912$, $b = 468{\cdot}024$, $c = 345{\cdot}678$.

(iv) $a = 1479{\cdot}48$, $b = 2465{\cdot}80$, $c = 3452{\cdot}12$.

It is easy to check the work by calculating C independently of A and B, and then finding the sum of A, B, and C. If this differs considerably from 180° we infer some error.

116. On account of the importance of applying checks we add the following examples of various methods which may be employed.

(1) Test the accuracy of the result of I. (a) (i), p. 133, by means of the formula—

$$\tan\frac{B-A}{2} = \tan(45-\phi)\tan\frac{A+B}{2}$$

$$\text{when } \tan\phi = \frac{a}{b}.$$

$$\begin{aligned} \log a &= 2{\cdot}275311 \\ \log b &= \underline{2{\cdot}365094} \\ \log\tan\phi &= \bar{1}{\cdot}910217 \\ \phi &= 39^\circ\ 7'\ 9''{\cdot}3 \\ 45-\phi &= 5^\circ\ 52'\ 50''{\cdot}7 \\ \log\tan 5^\circ\ 52'\ 50''{\cdot}7 &= \bar{1}{\cdot}012838 \\ \log\tan 47^\circ\ 36'\ 11'' &= \underline{\ {\cdot}039516} \\ \log\tan\frac{B-A}{2} &= \bar{1}{\cdot}052354 \\ \frac{B-A}{2} &= 6^\circ\ 26'\ 11'' \end{aligned} \qquad \begin{aligned} A &= 41^\circ\ 10' \\ B &= 54^\circ\ 2'\ 22'' \\ A+B &= 95^\circ\ 12'\ 22'' \\ \frac{B+A}{2} &= 47^\circ\ 36'\ 11'' \\ \frac{B-A}{2} &= 6^\circ\ 26'\ 11'' \end{aligned}$$

Ex. (176) Test the accuracy of the result of I. (a) (ii) by means of the above formula, by finding—

$$(1)\ \frac{B-A}{2},\quad (2)\ \frac{C-A}{2}.$$

Ex. (177) Test the accuracy of each of the results of I. (β) (i) by means of the above formula.

(2) Test the accuracy of the results I. (a) (ii), on p. 134, by means of the formula $b^2 - a^2 = 2Rc\sin(B-A)$.

$$\begin{aligned} b &= 77{\cdot}0876 \\ a &= \underline{54{\cdot}9934} \\ b+a &= 132{\cdot}081 \\ b-a &= 22{\cdot}0942 \end{aligned}$$

$$\begin{aligned} \log 2R &= 1{\cdot}921480 \\ \log c &= \underline{1{\cdot}897973} \\ \log 2Rc &= \underline{3{\cdot}819453} \\ \text{colog} &= \bar{4}{\cdot}180547 \\ \log 132{\cdot}081 &= 2{\cdot}120840 \\ \log 22{\cdot}0942 &= \underline{1{\cdot}344278} \\ \log\sin(B-A) &= \bar{1}{\cdot}645665 \\ B-A &= 26^\circ\ 14'\ 50''{\cdot}5 \end{aligned} \qquad \begin{aligned} B &= 67^\circ\ 27'\ 53'' \\ A &= \underline{41^\circ\ 13'\ \ 2''} \\ B-A &= 26^\circ\ 14'\ 51'' \end{aligned}$$

Ex. (178) Test the accuracy of the same results by means of the same formula, finding C − A.

(3) To test the accuracy of the results I. (β) (ii) by the special formula ($a_1 a_2 = b^2 - c^2$) for the ambiguous case.

$$\begin{aligned} b &= 77{\cdot}041 \\ c &= \underline{55\phantom{{\cdot}041}} \\ b + c &= 132{\cdot}041 \\ b - c &= 22{\cdot}041 \end{aligned}$$

$$\begin{aligned} \log 132{\cdot}041 &= 2{\cdot}120709 \\ \log 22{\cdot}041 &= \underline{1{\cdot}343231} \\ \log (b^2 - c^2) &= 3{\cdot}463940 \end{aligned} \qquad \begin{aligned} \log a_1 &= 1{\cdot}898180 \\ \log a_2 &= \underline{1{\cdot}565762} \\ \log a_1\, a_2 &= 3{\cdot}463942 \end{aligned}$$

Ex. (179) Test the accuracy of the answers to Ex. 169 by means of the above formula.

(4) To test the accuracy of the results II. (i), p. 136, by means of the formula—

$$\frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

$$\begin{aligned} \log 3428{\cdot}43 &= 3{\cdot}535095 \\ \log \sin 57^\circ\, 53'\, 17'' &= \underline{\bar{1}{\cdot}927889} \\ \log 2R &= 3{\cdot}607206 \end{aligned} \qquad \begin{aligned} \log 3277{\cdot}63 &= 3{\cdot}515560 \\ \log \sin 54^\circ\, 4'\, 19'' &= \underline{\bar{1}{\cdot}908354} \\ \log 2R &= 3{\cdot}607206 \end{aligned}$$

Ex. (180) Test the accuracy of the results II. (iii), p. 138, by means of the same formula.

117. By the kindness of Prof. A. Lodge, the author is enabled to give here a complete scheme of the method adopted at Cooper's Hill for the solution of a triangle when the three sides are given, and of the checks applied in the progress of the work. Some valuable remarks on logarithmic calculation are added.

$s = 479{\cdot}5830$		$\log r = 1{\cdot}939302$		
$a = 375{\cdot}3259$	$s - a = 104{\cdot}2571$	$\log (s - a) = 2{\cdot}018106$	$\log \tan \frac{1}{2}A = \bar{1}{\cdot}921196$	$\frac{1}{2}A = 39^\circ\ 49'\ 48{\cdot}2''$
$b = 312{\cdot}7715$	$s - b = 166{\cdot}8115$	$\log (s - b) = 2{\cdot}222226$	$\log \tan \frac{1}{2}B = \bar{1}{\cdot}717076$	$\frac{1}{2}B = 27\ \ 31\ \ 56{\cdot}7$
$c = 271{\cdot}0686$	$s - c = 208{\cdot}5144$	$\log (s - c) = 2{\cdot}319136$	$\log \tan \frac{1}{2}C = \bar{1}{\cdot}620166$	$\frac{1}{2}C = 22\ \ 38\ \ 15{\cdot}1$
$2s = 959{\cdot}1660$	$s = 479{\cdot}5830$	$\operatorname{colog} s = \bar{3}{\cdot}319136$	$\log s = 2{\cdot}680864$	$90^\circ\ \ 0'\ \ 0''$
	$\log s = 2{\cdot}680864$	$2 \log r = 3{\cdot}878604$	$\log r = 1{\cdot}939302$	

In the first column the value of s is found and placed at the top of the column to facilitate the subtractions required for the second column. These subtractions can be most easily performed by subtracting three or four figures mentally and then writing them down, and finishing the subtraction in the same way. It is less fatiguing than subtracting and writing down figure by figure. The first check is obtained in the second column by adding together $s - a$, $s - b$, $s - c$, the sum of which should be s. The third column is of the logarithms of the quantities in the second column with the exception of s, whose colog is required to obtain the value of r, since $r^2 = (s - a)(s - b)(s - c) \div s$. It is convenient to write log s at the foot of the second column for reference, and then to put its complement in the third column. The proportional parts should all be calculated mentally, and the correct logarithms written down at once. The obtained value of log r is placed at the head of the column to facilitate the subtractions required for the next column, which depends on the formulæ $\tan \frac{1}{2}A = r \div (s - a)$, etc. The work is checked in the fourth column by inserting log s and adding: the result should be log r, as it can be easily proved that $r = s \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C$. This checks the whole of the arithmetical work, including the finding of colog s from

log s, but does not check the correctness of the logarithms themselves. Finally, the angles are found and placed in the fifth column, the proportional parts being done mentally if such a book as Bremiker is used. If these angles do not add up to 90°, within two or three tenths of a second, there will be an error which must arise either in the fifth column itself, or in the finding of the logarithms in the third column.

Ex. (181). Given $a = 906{\cdot}9844$, $b = 194{\cdot}4227$, $c = 807{\cdot}7936$. Find A, B, C.

APPROXIMATE SOLUTION OF EQUATIONS.

Approximate solution of Equations by means of Trigonometrical Tables. The coefficients p and q are supposed positive.

118. *Quadratics.*

(i) $x^2 + px = q$.

Put $\tan\theta = \dfrac{2}{p}\sqrt{q}$,

then the two values of x are—

$$\tan\frac{\theta}{2}\sqrt{q} \text{ and } -\cot\frac{\theta}{2}\sqrt{q}.$$

Ex. $x^2 + 121{\cdot}807x = 385{\cdot}455$.

$$\begin{aligned}
\log 385{\cdot}455 &= \underline{2{\cdot}585974} \\
&\ 1{\cdot}292987 \\
\log 2 &= \ {\cdot}301030 \\
\operatorname{colog} 121{\cdot}807 &= \underline{\bar{3}{\cdot}914328} \\
\log\tan 17^\circ\, 52'\, 3'' &= \underline{\bar{1}{\cdot}508345} \\
\log\tan 8^\circ\, 56'\, 1''{\cdot}5 &= \bar{1}{\cdot}196451 \\
&\ \underline{1{\cdot}292987} \\
\log\left(\tan\frac{\theta}{2}\sqrt{q}\right) &= \ {\cdot}489438 \text{ (by addition)} \\
\log\left(\cot\frac{\theta}{2}\sqrt{q}\right) &= 2{\cdot}096536 \text{ (by subtraction)}
\end{aligned}$$

Hence the roots are $3{\cdot}0863$ and $-124{\cdot}892$.

Ex. (182) $x^2 + 36{\cdot}542x = 34{\cdot}691.$

(ii) $x^2 - px + q = 0.$

Put $\tan\theta = \frac{2}{p}\sqrt{q},$

then the two values of x are—

$$\cot\frac{\theta}{2}\sqrt{q} \text{ and } -\tan\frac{\theta}{2}\sqrt{q}.$$

E.g., the roots of $x^2 - 121{\cdot}807x = 385{\cdot}455$ are—

$$124{\cdot}892 \text{ and } -3{\cdot}0863.$$

Ex. (183) $x^2 - 3508x = 57129.$

(iii) $x^2 + px = -q\ (p^2 > 4q).$

Put $\sin\theta = \frac{2}{p}\sqrt{q},$

then the two values of x are—

$$-\tan\frac{\theta}{2}\sqrt{q} \text{ and } -\cot\frac{\theta}{2}\sqrt{q},$$

which may also be written—

$$-p\sin^2\frac{\theta}{2} \text{ and } -p\cos^2\frac{\theta}{2}.$$

Ex. (184) $x^2 + 171{\cdot}93x = -12375.$

(iv) $x^2 - px = -q\ (p^2 > 4q).$

Put $\sin\theta = \frac{2}{p}\sqrt{q},$

then the two values of x are—

$$\tan\frac{\theta}{2}\sqrt{q} \text{ and } \cot\frac{\theta}{2}\sqrt{q},$$

which may also be written—

$$p\sin^2\frac{\theta}{2} \text{ and } p\cos^2\frac{\theta}{2}.$$

Ex. (185) $x^2 - 503{\cdot}26x = -4871{\cdot}29.$

Cubics.

(i) $x^3 - px + q = 0\ (4p^3 > \text{or} = 27q^2).$

Put $\sin 3\theta = \frac{3q}{p} \div 2\sqrt{\frac{p}{3}},$

then the three values of x are—

$$\sin\theta \times 2\sqrt{\frac{p}{3}},\ \sin(60^\circ - \theta)\, 2\sqrt{\frac{p}{3}},\ -\sin(60^\circ + \theta) 2\sqrt{\frac{p}{3}};$$

the last may be also written $-\sin(120^\circ - \theta)\, 2\sqrt{\frac{p}{3}}$.

Ex. Solve the equation—

$$x^3 - 35x + 55{\cdot}5608 = 0.$$

$$\sin 3\theta = \frac{166{\cdot}6824}{35} \div 2\sqrt{\frac{35}{3}}$$

log 35	1·544068
log 3	·477121
	1·066947
	·533474
log 2	·301030
log $2\sqrt{\frac{35}{3}}$	·834504
colog	$\bar{1}$·165496
colog 35	$\bar{2}$·455932
log 166·6824	2·221889
log sin 44° 11′ 51″	$\bar{1}$·843317
log sin 14° 43′ 57″	$\bar{1}$·405358
	·834504
log <u>1·73725</u>	·239862
log sin 45 16′ 3″	$\bar{1}$·851503
	·834504
log <u>4·85297</u>	·686007
log sin 74 43′ 57″	$\bar{1}$·984395
	·834504
log <u>6·59020</u>	·818899

Hence the three values are 1·73725, 4·85297 and − 6·59020; on adding these to check the calculations, the sum which should be zero is found to be ·00002.

Ex. (186) (i) $x^3 - 21x + 7 = 0$.

(ii) $x^3 - 12x + 15 = 0$.

(iii) $x^3 - 7x + 7 = 0$.

(iv) $x^3 - 12x^2 + 41x - 29 = 0$.

(Reduce to standard form by writing $y + 4$ for x.)

Since by the Theory of Equations—

$$1{\cdot}73725 \times 4{\cdot}85297 \times 6{\cdot}5902$$

should be equal to 55·5608, we have a useful check on the accuracy of work by adding up the logarithms of these numbers. The sum is found to be 1·744768, which is, as it should be, log 55·5608.

(ii) $x^3 - px - q = 0$ ($4p^3 >$ or $= 27q^2$).

Put $\sin 3\theta = \dfrac{3q}{p} \div 2\sqrt{\dfrac{p}{3}}$,

then the three values of x are—

$$-\sin\theta \times 2\sqrt{\frac{p}{3}},\ -\sin(60^\circ - \theta) \times 2\sqrt{\frac{p}{3}},$$

$\sin\ (60^\circ + \theta) \times 2\sqrt{\dfrac{p}{3}}$; the last may also be written

$$\sin(120^\circ - \theta)\, 2\sqrt{\frac{p}{3}}.$$

E.g. The roots of $x^3 - 35x - 55{\cdot}5608 = 0$ are—

$$-1{\cdot}73725,\ -4{\cdot}85297 \text{ and } 6{\cdot}59020.$$

Ex. (187) $x^3 - 14{\cdot}58258x = 20{\cdot}06161$.

(iii) $x^3 - px + q = 0$ $(4p^3 < 27q^2)$.

Put $\sin\theta = \dfrac{p}{3q} \times 2\sqrt{\dfrac{p}{3}}$

$$\tan\phi = \sqrt[3]{\tan\frac{\theta}{2}}.$$

Then the only real root is $-\dfrac{2\sqrt{\dfrac{p}{3}}}{\sin 2\phi}$.

Ex. (188) $x^3 - 2x + 5 = 0$.

(iv) $x^3 - px - q = 0$ $(4p^3 < 27q^2)$.

Put $\sin\theta = \frac{p}{3q} \times 2\sqrt{\frac{p}{3}}$

$$\tan\phi = \sqrt[3]{\tan\frac{\theta}{2}}.$$

Then the only real root is $\dfrac{2\sqrt{\frac{p}{3}}}{\sin 2\phi}$.

Ex. (189) $x^3 - 3x - 7 = 0$.

(v.) $x^3 + px - q = 0$

Put $\tan\theta = \frac{p}{3q} \times 2\sqrt{\frac{p}{3}}$

$$\tan\phi = \sqrt[3]{\tan\frac{\theta}{2}}.$$

Then the only real root is $\cot 2\phi \times 2\sqrt{\frac{p}{3}}$.

Ex. (190) $x^3 + 7x = 504{\cdot}681$.

(vi.) $x^3 + px + q = 0$.

Put $\tan\theta = \frac{p}{3q} \times 2\sqrt{\frac{p}{3}}$

$$\tan\phi = \sqrt[3]{\tan\frac{\theta}{2}}.$$

Then the only real root is $-\cot 2\phi \times 2\sqrt{\frac{p}{3}}$.

Ex. (191) $x^3 + 100x + 1830 = 0$.

Ex. (192) Evaluate $[a^3(a + b)^2 \div b^4]^{\frac{1}{5}}$ when $a = 1{\cdot}41421$, $b = {\cdot}31831$.

Ex. (193) Solve the equation $a\cos\theta + b\sin\theta = c$ when $a = 2{\cdot}3026$, $b = 2{\cdot}71828$.

(Put $b = a\tan\alpha$; $c\cos\alpha = a\cos\beta$; $c\sin\alpha = b\cos\beta$; then $\theta = \alpha \pm \beta$.)

Ex. (194) Calculate x if (1) $\log_{10}x = \log_e 3$; (2) $\log_e x = \log_{10}3$.

Ex. (195) Adapt $x = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} - b\cos\alpha$ to logarithmic calculation, and evaluate when $a = 3b = 4{\cdot}77121$ $\alpha = 2°15'37''{\cdot}2$.

(Put $\tan\phi = 3\tan\alpha$; then $x = a\sin\alpha\tan\frac{1}{2}\phi$.)

Ex. (196) Find x from the equation $x^2 = \cot^2\alpha - \tan^2\alpha$ where $\alpha = 13°17'25''{\cdot}5$.

(Put $\sin\theta = \tan^2\alpha$, then $x = \cot\alpha\cos\theta = \tan\alpha\cot\theta$.)

Ex. (197) If
$$r = p/(1 + e\cos\phi)$$
$$r' = p/(1 + e\cos\phi')$$
$$r'' = p/(1 + e\cos\phi'')$$
express $u = \frac{1}{r}\sin(\phi'' - \phi) + \frac{1}{r'}\sin(\phi - \phi'') + \frac{1}{r''}\sin(\phi' - \phi)$ in a form adapted to logarithmic calculation, and evaluate it when $\phi = 17°4'$; $\phi' = 22°27'$; $\phi'' = 38°19'$; $p = 21$.

(On expansion the coefficient of e vanishes, and the remaining expression factorises. See *Mathematical Gazette*, No. 4.)

Ex. (198) From the pt. O three st. lines OA, OB, OC, are drawn in the same plane of length 1, 2, 3, respectively, and with the angles AOB, BOC, each equal to 60°. Find angle ABC.

Ex. (199) The longest side of a triangular plot of ground is 100 yds.; its perimeter is 250 yds.; and one angle is 40°. Find the remaining angles.

Ex. (200) Find the two roots of $x^2 = 20(x - 1)$ by the method (i) of § 81, (ii) of § 87, and show that the two results agree.

IV. PRACTICAL APPLICATIONS.

ACTUARIES' RULE.

119. For decimalising money and its applications, Percentages, Interest, Discount, etc.

In monetary calculations it is very convenient to be able to apply rapidly the actuaries' rule for writing down any sum of money expressed in shillings, pence and farthings, as a decimal of £1, *true to* 3 *places*, at sight.

The rule depends on the facts—

(1) That *a florin* is ·1 of £1.

(2) That *sixpence* is ·025 of £1, and hence that *a farthing* is only very slightly more than ·001 of £ (24 farthings = £·025). Hence any sum of money which is an exact multiple of sixpence can be given *exactly* as a decimal of £1 extending to three places; thus:—

(i) 6s. 6d. = £·325.

(The 3 for the 3 *florins*, the 25 for the *extra sixpence*.)

(ii) 17s. 6d. = £·875.

(The 8 for the 8 *florins*, the 75 for the *extra* 3 *sixpences*.)

Again 1, 2, 3 . . . 11 farthings expressed as decimals of £1 *true to* 3 *places* would be respectively ·001, ·002, ·003 . . . ·011, while 13, 14, 15 . . . 24 farthings would be ·014, ·015, ·016 . . . ·025 respectively.

Hence, *to* 3 *places*—

(iii) 13s. $7\frac{1}{4}$d. = £·680.

(The 6 for the 6 *florins*, the 80 made up of 75 + 5 for 3 sixpences + 5 farthings.)

(iv) 9s. $4\frac{1}{2}$d. = £·469.

(The 4 for the 4 *florins*, the 69 made up of 50 + 19 for 2 *sixpences* + 18 *farthings*.)

Since 3d. = ·0125 of £1, any sum of money which is an exact multiple of 3d. can be given *exactly* as a decimal of £1, extending to 4 places; thus:—

(v) 3s. 9d. = ·1875.

(The 1 for *the florin*, the 87½ made up of 75 + 12½, for 3 *sixpences* + 3d.)

Conversely any three place decimal of £1 may be read off in shillings, etc.; thus:—

(vi) £·275 = 5s. 6d.

(vii) £·839 = 16s. 9¼d.

In connection with the above rule it may be noted that the identities—

$$1 \text{ farthing} = £\cdot001\,(1 + \tfrac{1}{24})$$
$$£\cdot001 = 1 \text{ farthing}\,(1 - \cdot04)$$

may be utilised for the reduction of farthings to £ s. d., or *vice versâ*: thus:—

(viii) 78350219 farthings	24) 78350219
	651425
= £81614·811	1112
	3264592
= £81614 16s. 2¾d.	81614·811

(ix.) £567 3s. 10¼d.	567193
= £567·193	22688
= 544505 farthings.	544505

The preceding verbal rules for the approximate decimalisation of money may be dispensed with if the following table be committed to memory:—

s. d.		£
2 0	=	·100
1 6	=	·075
1 0	=	·050
6	=	·025
¼	=	$\cdot001\frac{1}{24}$ = $\cdot00104\frac{1}{6}$

Thus:—

(x) 9s. 4½d. = ·450
+ $\cdot018\frac{18}{24}$
= ·46875

It is easy to carry the approximation to a higher degree of accuracy than the verbal rules give; in fact, to any required degree of accuracy; thus:—

$$\begin{aligned}
& & & \pounds \\
&\text{(xi) } 13\text{s. } 7\tfrac{1}{4}\text{d.} &=\ & \cdot 675 \\
& &+\ & \cdot 005\tfrac{5}{24} \\
& &=\ & \cdot 680208\dot{3} \text{ (since } \tfrac{5}{24} = \tfrac{2\cdot 5}{12}\text{)}
\end{aligned}$$

or thus:—

$$\begin{aligned}
\text{(xii) since } 1\,f &= \cdot 00104\tfrac{1}{6} \\
7\,f &= \cdot 00729\tfrac{1}{6} \\
&= \cdot 007291666\ldots
\end{aligned}$$

Working from *left to right*, we write down ·007 at once: then get seven times four 28 and increase the 28 by the $1\frac{1}{6}$. A little practice will enable students to do this with great ease.

As an additional example 17s. $9\frac{1}{2}$d. $= \cdot 88958\frac{1}{3}$
$= \cdot 88958333\ldots$

Thus the decimal required may be written down to any required degree of accuracy.

Ex. To reduce £7556 4s. $3\frac{1}{2}$d. to the decimal of £9872 2s. $9\frac{1}{4}$d.

$$\frac{\pounds 7556 \text{ 4s. } 3\frac{1}{2}\text{d.}}{\pounds 9872 \text{ 2s. } 9\frac{1}{4}\text{d.}} = \frac{7556\cdot 215}{9872\cdot 139} = \cdot 76541$$

$$\begin{array}{r|l}
 & \underline{\cdot 76541} \\
987213\cdot 9\ ) & 755621\cdot 5 \\
\cdots\cdot & 64571\cdot 8 \\
 & 5339\cdot 0 \\
 & 402\cdot 9 \\
 & 8\cdot 0
\end{array}$$

Ex. To find what dividend can be paid by a bankrupt whose assets are £389 15s. $4\frac{3}{4}$d. and liabilities £987 6s. $9\frac{1}{2}$d.

$$\frac{\pounds 389 \text{ 15s. } 4\frac{3}{4}\text{d.}}{\pounds 987 \text{ 6s. } 9\frac{1}{2}\text{d.}} \text{ of } \pounds 1 = \frac{389\cdot 770}{987\cdot 340} \text{ of } \pounds 1 = \pounds\cdot 395 = 7\text{s. } 10\tfrac{3}{4}\text{d.}$$

$$\begin{array}{r|l}
 & \cdot 395 \\
9873\cdot 4\ ) & 3897\cdot 7 \\
\cdots\cdot & 935\cdot 7 \\
 & 47\cdot 1
\end{array}$$

Ex. Goods bought for £65 8s. $2\frac{3}{4}$d. are sold for £63 7s. $4\frac{1}{4}$d.; to find the loss per cent. on the buying price.

$$\text{Loss per cent.} = \frac{204\cdot4}{65\cdot412} \qquad \begin{array}{r} 65\cdot412 \\ 63\cdot368 \\ \hline 2\cdot044 \end{array} \qquad \begin{array}{r} 3\cdot12 \\ 654\backslash 1\backslash\cdot2\ )\ 20440 \\ 816 \\ 162 \end{array}$$

$$= 3\cdot12$$

Ex. To divide £314 17s. $2\frac{3}{4}$d. as nearly as possible in the proportions of $3\frac{1}{3}$, $5\frac{1}{9}$, $6\frac{2}{7}$.

$$\begin{array}{r} 3\cdot33333 \\ 5\cdot11111 \\ 6\cdot28571 \\ \hline 14\cdot7302 \end{array} \qquad \begin{array}{r} 21\cdot3752 \\ \hline 14\backslash7\backslash3\backslash0\backslash2\ )\ 3148610 \\ 202570 \\ 55268 \\ 11077 \\ 766 \\ 29 \end{array}$$

$$\text{1st share} = £314\cdot861 \times \frac{3\frac{1}{3}}{14\cdot7302} \qquad \begin{array}{r} 21\cdot3752 \\ \hline 64\cdot1256 \\ 7\cdot1251 \\ \hline 71\cdot2507 \end{array}$$

$$= £21\cdot3752 \times 3\tfrac{1}{3}$$

$$= £71\ 5\text{s. } 0\tfrac{1}{4}\text{d.} -$$

$$\text{2nd share} \doteq £21\cdot3752 \times 5\tfrac{1}{9} \qquad \begin{array}{r} 21\cdot3752 \\ \hline 106\cdot8760 \\ 2\cdot3750 \\ \hline 109\cdot2510 \end{array}$$

$$= £109\ 5\text{s. } 0\tfrac{1}{4}\text{d.}$$

$$\text{3rd share} = £21\cdot3752 \times 6\tfrac{2}{7} \qquad \begin{array}{r} 21\cdot3752 \\ \hline 128\cdot2512 \\ 3\cdot0536 \\ \hline 134\cdot3584 \end{array}$$

$$= £134\ 7\text{s. } 2\text{d.} +$$

To find the Simple Interest on £549 7s. $8\frac{3}{4}$d. for 3 yrs. 7 mths. at $4\frac{3}{8}$ p. c. per ann.

(i) By ordinary arithmetic; (ii) by logarithms.

(i)

$$\begin{array}{rl}
 & 5{\cdot}49386 \\
 & \underline{4\tfrac{3}{8}} \\
 & 21{\cdot}97544 \\
\tfrac{1}{8} & \underline{\cdot 68673\tfrac{1}{4}}\ (\times 3) \\
 & 24{\cdot}03564 \\
 & \underline{3\tfrac{7}{12}} \\
 & 72{\cdot}10692 \\
\tfrac{6}{12} & 12{\cdot}01782 \\
\tfrac{1}{12} & \underline{2{\cdot}00297} \\
 & 86{\cdot}128 \\
 & £86\ 2s.\ 6\tfrac{3}{4}d.
\end{array}$$

(ii)

$$\begin{array}{rr}
549{\cdot}386 & 2{\cdot}739877 \\
\cdot 04375 & \bar{2}{\cdot}640978 \\
3{\cdot}58333 & \underline{\cdot 554287} \\
86{\cdot}128 & 1{\cdot}935142
\end{array}$$

£86 2s. 6¾d.

When the fraction of a year is given as a number of days it will be better to express it as a decimal. The decimalising may be effected or checked by means of a table such as that given on p. 184. It should be remembered that periods of 73, 146, 219, 292 days are finite decimals, being ·2, ·4, ·6, ·8 of a year respectively.

Ex. To find the Simple Interest on £985 2s. 7d. at 2¾ p. c. per ann. for 5 yrs. 127 dys.

(i) By ordinary arithmetic—

$$\begin{array}{rl}
\tfrac{1}{2} & 9{\cdot}85129 \\
 & \underline{2\tfrac{3}{4}} \\
 & 19{\cdot}70258 \\
\tfrac{1}{2} & 4{\cdot}92565 \\
 & \underline{2{\cdot}46282} \\
 & 27{\cdot}09105 \\
 & \underline{5{\cdot}347945} \\
 & 135{\cdot}45525 \\
 & 8{\cdot}12732 \\
 & 1{\cdot}08364 \\
 & \cdot 18964 \\
 & \cdot 02438 \\
 & \cdot 00108 \\
 & \underline{\cdot 00014} \\
 & 144{\cdot}881
\end{array}$$

$$\begin{array}{rr}
365\ ) & 127{\cdot}00000 \\
 & \cdot 50550 \\
 & 7{\cdot}9469 \\
 & \underline{12{\cdot}311} \\
 & \underline{\cdot 347945}
\end{array}$$

$$\begin{array}{rl}
 & \text{days}\quad \text{yr.} \\
\text{or thus} & 146 = \underline{\cdot 4} \\
 & 18\tfrac{1}{4} = \cdot 05 \\
\text{by table} & \tfrac{3}{4} = \underline{\cdot 002055} \\
 & 127 = \cdot 347945
\end{array}$$

(ii) By logarithms—

9·85129	·9934891
	40
2·75	·4393327
5·34794	·7281833
	32
144·8813	2·1610123
	84
	300) 39 (13

£144 17s. 7½d.

Ex. To find the amount of £468 9s. 7d. at Compound Interest in 5 years at 4 p. c.

(i) By ordinary arithmetic—

	£
£468 9s. 6d. =	468·475
1d. =	·004167
	468·47,92
	18·7392
	487·21,84
	19·4887
	506·70,71
	20·2683
	526·97,54
	21·0790
	548·05,44
	21·9222
	569·9766

£569 19s. 6½d. (to the nearest farthing).
£569 19s. 6d. (to the nearest penny).

The work might be abbreviated by performing the addition and multiplication simultaneously.

(ii) By logarithms—

1·04	·01703334	(8 fig. table)
$(1{\cdot}04)^5$	·0851667	
468·47	2·6706818	
9	83	
2	2	
569·97	2·7558570	
	20	
7	50	

For the use of computers who have much to do with questions of Compound Interest tables have been constructed showing the amount of £1 for 1, 2, 3, . . . 100 years at various percentages. From such a table we find that the amount of £1 for 5 years at 4 p. c. per ann., *true to 6 figures*, is £1·21665.

If we are satisfied with correctness to the nearest penny we might work the above example by one piece of contracted multiplication, thus:—

```
468·4792
  1·21665
---------
468·4792
 93·6958
  4·6848
  2·8108
   ·2811
   ·0234
---------
569·975
£569 19s. 6d.
```

Tables have also been constructed showing what sum would amount to £1 in 1, 2, 3 . . . 100 years at various percentages. Such a sum is called the **Present Worth** of £1. On reference we find the Present Worth of £1 due at the end of 5 years, Compound Interest being reckoned at 4

p. c., to be £·82193. Since this must clearly be, if correct, identical with $\frac{1}{1{\cdot}21665}$, we may also work the same example by means of a single piece of contracted division.

$$\begin{array}{rl} & \underline{569{\cdot}976} \\ 82193\ ) & 46847917 \\ & \ \ 57514 \\ & \ \ \ \ 8198 \\ & \ \ \ \ \ \ 801 \\ & \ \ \ \ \ \ \ \ 61 \\ & \ \ \ \ \ \ \ \ \ \ 5 \end{array}$$

the last figure being doubtful.

Ex. To find the Present Worth of £537 13s. 5¾d. due in 4 yrs. 4 mths., Simple Interest being reckoned at 4½ p. c. per ann.

$$\begin{array}{rrr} \text{(i)}\ £537\ 13\text{s}.\ 6\text{d}. = & £537{\cdot}675 & 4{\cdot}5 \\ \tfrac{1}{4} = & \underline{{\cdot}00104} & \underline{4} \\ & \underline{537{\cdot}67396} & 18 \\ 1195\ )\ 537673{\cdot}96 & & \underline{1{\cdot}5} \\ 678{\cdot}41 & & 19{\cdot}5 \\ 9813{\cdot}64 & & \frac{100}{119{\cdot}5} = \frac{1000}{1195}. \\ 51147{\cdot}4 & & \\ \underline{11\qquad\qquad} & & \\ 449{\cdot}9364 & & \\ £449\ 18\text{s}.\ 8\tfrac{3}{4}\text{d}. & & \end{array}$$

$$\begin{array}{lrr} \text{(ii)} & 537{\cdot}67 & 2{\cdot}7305158 \\ & 4 & \underline{32} \\ & & 2{\cdot}7305190 \\ & 1{\cdot}195 & \underline{{\cdot}0773679} \\ & 449{\cdot}93 & 2{\cdot}6531511 \\ & 6 & \underline{450} \\ & & 61 \\ & & £449\ 18\text{s}.\ 8\tfrac{3}{4}\text{d}. \end{array}$$

Ex. To find to the nearest penny the Present Worth of £572 6s. 7d. due in 5 years, Compound Interest being reckoned at 6 p. c. per ann.

(i)

1·06)	572·329
	2·585
	40·93
	1
1·06)	539·933
	9 95
	·37
1·06)	509·371
	5· 51
	8 4
1·06)	480·539
	6·551
	53·34
1·06)	453·339
	9·113
	28·78
	427·679

(ii)

572·32	2·7576389
9	69
	2·7576458
1·06	·02530587 (× 5)
427·67	2·6311165
8	1088
	77

£427 13s. 7d. (to nearest penny)

£427 13s. 7d.

The above might be solved by a piece of contracted multiplication or division if the same tables are consulted. Compare pp. 155, 156.

(i) By multiplication, finding by table that Present Worth of £1 in 5 yrs. at 6 p. c. per ann. = £·74726.

(ii) By division, finding by table that amount of £1 in 5 yrs. at 6 p. c. is 1·3382255.

(i)	(ii)
572·329	427·679
·74726	133822·55) 57232917
400·6303	37039
22·8932	10274
4·0063	907
·1145	104
·0343	11
427·679	

Ex. To find the Present Value of an annuity of £572 6s. 7d. for 5 yrs., Compound Interest being reckoned at 6 per cent. per ann.

(i) As on p. 157, we might find the Present Values of the five successive sums to be—

539·933
509·371
480·539
453·339
427·679

Adding 2410·861 is found to be that of the annuity.

(ii) To 8 decimal places colog 1·06 = $\bar{1}$·97469413, writing down this and its multiples as far as required and then adding numbers corresponding to these logarithms in the tables, thus:—

·943395	$\bar{1}$·9746941
·889997	$\bar{1}$·9493883
·839620	$\bar{1}$·9240824
·792094	$\bar{1}$·8987765
·747258	$\bar{1}$·8734707
4·21236	·6245255
572·329	2·7576458
2410·8	3·3821713
6	612
2410·86	101

(iii) By multiplication; a table giving £4·21236 the Present Value of an annuity of £1 for 5 years at 6 per cent.

(iv) By division; a table giving £·23740 as the annuity for 5 years, with interest at 6 p. c. per ann. which £1 will purchase.

(iii)	(iv)
572·3292	2410·8
4·21236	2374·0) 5723292
2289·3168	9752
114·4658	256
5·7233	19
1·1447	
·1717	
·0343	
£2410·857	

The closeness of the approximation of course depends on that of the tables used. We can obtain about the same number of figures correct by our multiplication or division as we find in the tables. No advantage would be gained by working either of them by the uncontracted methods.

If an electric horse-power cost £15 per annum for 5 hrs. per night, what is the value of 1 watt-hour?

1 horse-power = 746 watt.

$$\text{Cost} = \frac{£15}{746 \times 5 \times 365} = \frac{£3}{746 \times 365} = \frac{£·6}{746 \times 73} = £·000011 = ·001 \text{ of a farthing}$$

73) ·600
647
116
746) ·008219
5
7
·000011

120. Solutions of Various Physical and other Problems, involving Approximate Multiplication and Division, Involution, etc., and various Artifices of Reduction.

(*a*) Without logarithms.

(i) What will be resistance of $3\frac{1}{2}$ miles of wire $\frac{1}{12}$th of a sq. in. in section, made of German silver, at a temperature of 37·5° (P.E.)?

By formula in Ayrton, resistance =

$$(1 + \cdot000443t + \cdot000000152t^2)\,\frac{8\cdot24}{10^6} \times \frac{3\frac{1}{2} \times 5280 \times 12}{\frac{1}{12}}.$$

Note that $37\cdot5 = \frac{300}{8}$.

·000443	·000000152	5280
8) ·1329	·0000456	15840
·0166125	8) ·01368	2640
	8) ·00171	18480
	·00021375	221760
	1·0166125	2·661120
	1·01683	8·24
		21·28896
		·53222
		·10644
		21·9276
		1·01683
		21·9276
		·2193
		·1316
		175
		7
		22·30

(ii) A Daniell's battery produces a deflection of 38° on a tangt. galvanometer when a resistance of 27 ohms is inserted in the circuit, and a deflection of 46° when the

resistance is reduced to 12 ohms. What is the resistance of the battery if that of the galvanometer is 2·5 ohms?

By the formula, resistance =

$$\frac{\tan 38^\circ\,(27 + 2{\cdot}5) - \tan 46^\circ\,(12 + 2{\cdot}5)}{\tan 46^\circ - \tan 38^\circ}$$

By an algebraical transformation we may write this—

$$\frac{15 \tan 46^\circ - 29{\cdot}5\,(\tan 46^\circ - \tan 38^\circ)}{\tan 46^\circ - \tan 38^\circ}$$

$$= \frac{15 \tan 46^\circ}{\tan 46^\circ - \tan 38^\circ} - 29{\cdot}5$$

$$= \frac{15{\cdot}533}{{\cdot}2542} - 29{\cdot}5$$

$$= 31{\cdot}6$$

1·0355	10·355		
·7813	5·178		
·2542	15·533		
2542	) 155330 (	61·1	
	281	29·5	
	27	31·6	

(iii) If in the weight electrometer (p. 88, P.E.) the suspended plate B were square and its edge 1·4 centimetres long, and if the distance between it and the fixed plate A were 3 millimetres, what potential difference in volts must be maintained between A and B so that the attractive force may be 1 milligramme?

$$V = \frac{{\cdot}3 \times 10^5}{1{\cdot}4}\sqrt{\frac{{\cdot}001}{4{\cdot}508}}$$

$$= \frac{30000}{1{\cdot}4}\sqrt{\frac{{\cdot}01}{45{\cdot}08}}$$

$$= \frac{3000}{1{\cdot}4}\;\frac{1}{\sqrt{45{\cdot}08}}$$

$$= 319{\cdot}2$$

	6·7141
	45·08
127	9·08
1341	1900
1342	559
	22
	6·7141
	2·6856
	9·3997
94)	30000
	864
	181
	319·2

(iv) In the same instrument if the suspended plate be circular, what must be its diameter so that, when at a distance of one millimetre from the fixed plate, a difference of potentials of 10 volts shall produce an attraction of ·01 gramme (P.E., p. 90)?

$$\text{Diameter} = 2 \times \dot{\cdot 1} \times 10^4 \sqrt{\frac{\cdot 01}{\pi \times 4{\cdot}508}}$$

$$= \frac{200}{\sqrt{\pi \times 4{\cdot}508}}$$

$$= \frac{200}{3{\cdot}763}$$

$$= 53{\cdot}1$$

```
          4·508
          3·1416
         -------
         13·524
            ·451
            ·180
               5
               2
         -------
         14·16 ( 3·763
    67    5·16
   746     4700
   75·2     224
          53·15
  3763 ) 200000
          11850
            561
            185
```

(v) How many amperes would deflect the needle of a tangent galvanometer 60° in the year 1886, the controlling force being the horizontal component of the earth's magnetism, and the galvanometer having a bobbin five inches in radius, wound with six convolutions of wire?

$$\text{No. of amperes} = \frac{\cdot 73735 \times 5 \times \sqrt{3}}{6}$$

$$= \frac{6{\cdot}386}{6}$$

$$= 1{\cdot}064$$

```
   ·73735
  --------
  3·68675
  1·73205
  --------
  3·68675
  2·58073
   ·11060
      737
       18
  --------
  6·386
```

(vi) The expression—

$$U = \frac{5\frac{2}{9}}{3}\{(193{\cdot}35 \times 13{\cdot}13) - (3{\cdot}35 \times 216) + (13{\cdot}4 \times 202{\cdot}87)\}$$

for the work done in inch-tons in compressing cotton, is taken from *Industries*, vol. vi., p. 578.

Find the work in inch-tons and in foot-lbs., and the H.-P. of an engine which will run the press up in 47·5 secs.

```
 193·35        216          202·8,7
2513·55        648         2637·31
  25·14        712·8       2718·46
2538·69      - 723·6     + 2538·69
                           5257·15
                            723·6
                           4533·5,5 × 47
                          181342
                        9 ) 213077
                         3 ) 23675
                             7892 inch-tons.
                        15784
                        173624
                   12 ) 17678080
                         1473173 foot-lbs.
```

1 inch-ton = $\frac{2240}{12}$ foot-lbs.

Note that we are entitled to put the foot-lbs. down as 1473000, for the result was obtained from an approximation 7892 true to 4 figs.

$$\text{H.-P.} = \frac{1473000}{550 \times 47{\cdot}5}$$
$$= \frac{1473000}{55 \times 475}$$
$$= \frac{1473000}{5 \times 11 \times 25 \times 19}$$
$$= \frac{11784}{209}$$
$$= 56{\cdot}4.$$

```
20,9 ) 11784
          30
        1,38
        56·4
```

(vii) The expression—

$$t = \frac{29084 - 2546}{(2{\cdot}294 \times {\cdot}712) + (5{\cdot}485 \times {\cdot}301) + (12{\cdot}854 \times {\cdot}222)}$$

for the temperature of the resultant gases after the combustion of water gas made from coke, is taken from *Industries*.

$$\begin{array}{rrrr}
2{\cdot}294 & 5{\cdot}485 & 12{\cdot}854 & 29084 \\
{\cdot}712 & {\cdot}301 & {\cdot}222 & 2546 \\
\hline
1{\cdot}6058 & 1{\cdot}6455 & 2{\cdot}5708 & 26538 \\
229 & 55 & {\cdot}2571 & \\
46 & 1{\cdot}6510 & {\cdot}0257 & \\
\hline
1{\cdot}6333 & & 2{\cdot}8536 & \\
 & & 1{\cdot}6510 & \\
 & & 1{\cdot}6333 & \\
\hline
 & & 6{\cdot}1379 & \\
 & & \underline{4324} & \\
\end{array}$$

$$\begin{array}{rr}
61{\cdot}37{\cdot}9\ ) & 26538000 \\
 & 1986 \\
t = 4324^\circ \qquad & 145 \\
 & 22 \\
\end{array}$$

(viii) The equation—

$$(33{\cdot}95 + \tfrac{14}{6}x)\,{\cdot}473 = 27{\cdot}72 - \tfrac{22}{6}x$$

for the quantity of carbon consumed under certain conditions in a blast furnace, is taken from *Industries*, vol. vii., p. 431.

$$33{\cdot}95 \times {\cdot}473 + ({\cdot}473 \times 2\tfrac{1}{3})x = 27{\cdot}72 - 3\tfrac{2}{3}x$$

$$\begin{array}{rr}
{\cdot}473 & 33{\cdot}95 \\
\hline
{\cdot}946 & {\cdot}473 \\
{\cdot}158 & 13{\cdot}580 \\
3{\cdot}667 & 2{\cdot}377 \\
\hline
4{\cdot}771x & {\cdot}102 \\
 & 16{\cdot}06 \\
 & 11{\cdot}66 \\
\end{array}$$

$$\begin{array}{rr}
x = & 2{\cdot}44 \\
477{\cdot}1\ ) & 1166 \\
 & 212 \\
 & 21 \\
\end{array}$$

(ix) The equation—

$$(10 - x)\,(6{\cdot}43)^2 = (2 - \tfrac{6\cdot43}{18\cdot65}x)\,(18{\cdot}65)^2$$

to determine the viscous friction of a fluid, is taken from *Industries*, vol. vii., p. 415.

$$x\{18{\cdot}65 \times 6{\cdot}43 - ({\cdot}643)^2\} = 2(18{\cdot}65)^2 - 10(6{\cdot}43)^2$$

$$\begin{array}{rllll}
18{\cdot}65 & 1 & 1 & 6 & 36 \\
\underline{6{\cdot}43} & 28 & 324 & 12{\cdot}4 & 40{\cdot}96 \\
111{\cdot}90 & 36{\cdot}6 & 345{\cdot}96 & 12{\cdot}8 & 41{\cdot}34 \\
7{\cdot}46 & & \underline{347{\cdot}79} & & \\
\underline{{\cdot}56} & & 695{\cdot}6 & & \\
119{\cdot}92 & & \underline{413{\cdot}4} & & \\
\underline{41{\cdot}34} & & 282{\cdot}2 & & \\
78{\cdot}58x & & & &
\end{array}$$

$$\begin{array}{rl}
x = & \underline{3{\cdot}6} \\
785{\cdot}8\ )\ & 2822 \\
& \ \ 465
\end{array}$$

(x) At what temperature will a wire $3\frac{1}{2}$ miles long, $\frac{1}{12}$ of a sq. in. in section, made of German silver, have a resistance of 22·23 ohms (P.E.)?

Taking from a table the resistance of an inch cube of German silver to be 8·24 microhms, we have to solve the quadratic—

$$(1 + {\cdot}000443t + {\cdot}000000152t^2)\,\frac{8{\cdot}24}{10^6} \times \frac{3{\cdot}5 \times 5280 \times 12}{\frac{1}{12}} = 22{\cdot}23$$

$$(1 + {\cdot}000443t + {\cdot}000000152t^2) \times 21{\cdot}9276 = 22{\cdot}23$$
$$1 + {\cdot}000443t + {\cdot}000000152t^2 = 1{\cdot}01379$$
$$443t + {\cdot}152t^2 = 13790$$

For a first approximation neglect the term $\cdot000000152t^2$.

$$(1 + \cdot000443t)\frac{8\cdot24}{10^6} \times \frac{3\cdot5 \times 5280 \times 12}{\frac{1}{12}} = 22\cdot23$$

$$\begin{array}{r} 5280 \\ 3\frac{1}{2} \\ \hline 15840 \\ 2640 \\ \hline 18480 \\ 12 \\ \hline 221760 \\ 12 \\ \hline 2\cdot661120 \\ 8\cdot24 \\ \hline 21\cdot28896 \\ \cdot53222 \\ \cdot10644 \\ \hline 21\cdot9276 \end{array}$$

$$1 + \cdot000443t = \frac{22\cdot23}{21\cdot9276}$$

$$= 1\cdot01379$$

$$443t = 13790$$

$$t = 31$$

$$\begin{array}{rl} & 1\cdot01379 \\ 219276\ ) & 222300 \\ \cdots\cdot & 3024 \\ & 831 \\ & 173 \\ & 20 \end{array} \qquad \begin{array}{rl} & 31 \\ 443\ ) & 13790 \\ & 500 \end{array}$$

For a second approximation in

$$443t + \cdot152t^2 = 13790$$

for t^2 write 961 from the first approximation.

$$443t = 13790 - 146$$

$$= 13644$$

$$t \quad 30\cdot8$$

$$\begin{array}{r} 961 \\ \hline 96\cdot1 \\ 48\cdot1 \\ 1\cdot9 \\ \hline 146 \\ 13790 \\ \hline \end{array}$$

$$\begin{array}{rl} 443\ ) & 13644 \\ & 5 \\ & 3 \\ \hline & 30\cdot8 \end{array}$$

(xi) To find a sector of a circle which is bisected by its chord.

Let ϕ be the angle of the sector and a the radius of the circle.

Then $a^2 \sin \phi = \frac{1}{2} a^2\phi$,

$\therefore\ 2 \sin \phi = \phi$.

Hence $\phi > \frac{\pi}{2}$.

Put $\phi = \pi - \theta$,
then $2 \sin \theta + \theta = \pi$.

Since $\sin \theta < \theta$ we have $\theta > \frac{\pi}{3}$. Hence we try $\theta = 70°$.

Using Bremiker's Tables, where we conveniently find the values of θ and $\sin \theta$ tabulated on the same page, we find $2 \sin \theta + \theta = 3{\cdot}10111$, which is too little. Trying 71° we get 3·13012, which is again too little, but much nearer.

Trying 71° 10′ we get 3·13501.

Trying 71° 20′ we get 3·13980, which is very near, but still too little.

Trying 71° 25′ we get 3·14218, which is too great.

Now 3·14159 lies between the two last values, and rather nearer the greater value.

Trying 71° 23′ we get 3·14123.

Trying 71° 24′ we get 3·14171.

Here ·00048 is the diff. for 60″.

Hence ·00036 is the diff. for 45″.

This satisfies to six significant figures.

Hence $\phi = 108° 36′ 15″$.

A more accurate solution by Euler gives 108° 36′ 13″·7575.

(xii) What is the safe twisting moment allowable in a shaft 5·375 in. in diam., allowing a working stress (s) of 3·25 tons per sq. in.?

If the radius of the shaft be $15\frac{5}{16}$, what will be the allowable weight?

Moment in inch-tons	$= \cdot 196 d^3 s$	5·375	·7304
	$= 98 \cdot 9$	$(5 \cdot 375)^3$	2·1912
		·196	$\bar{1}$·2923
		3·25	·5119
		98·9	1·9954
Allowable weight	$= \dfrac{98 \cdot 9}{15 \cdot 31}$	15·31	1·1850
		6·463	·8104

$$\begin{array}{r|l} & 6 \cdot 46 \\ 15 \cdot 31 & 989 \\ & 70 \\ = 6 \cdot 46 & 9 \end{array}$$

(xiii) A hydraulic jack has a ram 3″ in diameter and a pump plunger $\frac{5}{8}$ in. in diameter; the handle is 22″ long from fulcrum, and the pump plunger is connected with the handle at 3″ from the fulcrum. Find (i) to nearest lb. what load a man can lift when he presses on the handle with a force of 27 lbs.; (ii) to units, how many strokes of the handle, each of 9″ long, will be required to raise the ram 3″·5?

(i) $\dfrac{3^2 \times 22 \times 27}{(\frac{5}{8})^2 \times 3}$

$= \dfrac{66 \times 27 \times 64}{25}$

$= 4562$

198
198
1782
10692
1140·48
4562

(ii) $3 \cdot 5 \times \dfrac{3^2}{(\frac{5}{8})^2} \times \frac{22}{3} \times \frac{1}{9}$

$= 3 \cdot 5 \times \frac{64}{25} \times \frac{22}{3}$

$= 66$

64
192
32
2·24
8·96 ($\times 7\frac{1}{3}$)
62·72
2·99
66·

(*b*) With logarithms.

(i) What load will a cylindrical iron shaft 5·75 in. in diameter and 5 ft. 7·5 in. long between its supports carry, the load being at the centre of the shaft?

The formula is $w = \dfrac{\cdot 726 d^3 s}{l}$

where d is the diameter, l the length, and s the breaking stress of the material of the shaft, which we shall take as 8700 lbs. per sq. in. for the special sort of iron employed.

5·75	·7597
$(5\cdot75)^3$	2·2791
8700	3·9395
·726	$\bar{1}\cdot8609$
colog 5·625	$\bar{1}\cdot2516$
	5·3311
214300 lbs.	

(ii) From the formula $\dfrac{s_{n-r-1}}{s_{n-r}} = \frac{1}{4}\left\{\left(\dfrac{s_{n-r}}{s_{n-r+1}}\right)^4 + 3\right\}$

calculate (i) the ratios $\dfrac{s_{n-2}}{s_{n-1}}, \dfrac{s_{n-3}}{s_{n-2}}, \ldots \dfrac{s_{n-6}}{s_{n-5}}$

(ii) the ratios $\dfrac{s_{n-2}}{s_n}, \dfrac{s_{n-3}}{s_n}, \ldots \dfrac{s_{n-6}}{s_n}$

being given that $\dfrac{s_{n-1}}{s_n} = 1\cdot100$.

The ratios are connected with some investigations on the machinery required to work presses for baling cotton, etc.; s_0 being the depth in inches occupied by the bale in the box; $s_1, s_2, s_3, \ldots s_n$ the depths at which the pumps successively "knock off". *Industries*, vol. vi., p. 386.

1·100	·0414	1·100	·0414
1·464	·1656		·0478
1·116	·0478	1·228	·0892
1·553	·1912		·0561
1·138	·0561	1·397	·1453
1·677	·2244		·0679
1·169	·0679	1·634	·2132
1·869	·2716		·0852
1·217	·0852	1·988	·2984
2·192	·3408		·1134
1·298	·1134	2·581	·4118

Hence for (i) we have 1·116, 1·138, 1·169, 1·217, 1·298;
for (ii) we have 1·228, 1·397, 1·634, 1·988, 2·581.

(iii) Calculate from the formula $A = K\sqrt{D}$ the number of amperes in the current passing through a Siemens dynamometer; D being deflection in grades, K a constant depending on the strength of the spring and the number of convolutions in coils, and A the number of amperes—
(i) when $K_1 = 3{\cdot}201$; (ii) $K_2 = 11{\cdot}10$; for 10, 20, 30, . . . 60 grades of deflection.

Here $\log K_1 = {\cdot}5053$, $\log K_2 = 1{\cdot}0453$, and hence $\log K_2/K_1 = {\cdot}5400$, $\log A_2 = \log A_1 + {\cdot}54$.

D	10	20	30	40	50	60
log D	1·0000	1·3010	1·4771	1·6021	1·6990	1·7782
$\log\sqrt{D}$	·5000	·6505	·7386	·8011	·8495	·8891
$\log K_1$	·5053	·5053	·5053	·5053	·5053	·5053
$\log A_1$	1·0053	1·1558	1·2439	1·3064	1·3548	1·3944
A_1	10·12	14·32	17·54	20·25	22·63	24·78
$\log A_2$	1·5453	1·6958	1·7839	1·8464	1·8948	1·9344
A_2	35·10	49·64	60·80	70·21	78·48	85·98

Check by noticing that $K_2 = K_1 \times 3\frac{1}{2}$ nearly.

(iv) In Ayrton's *Practical Electricity* we find the resistance of a certain condenser given as—

$$r = \frac{\cdot 4343 \times 30 \times 3}{F \log \dfrac{K}{K'}} \text{ megohms}$$

where $F = \dfrac{173\cdot 6 \times 230}{295}$

and $\dfrac{K - K'}{K} \quad \dfrac{112 \times 295}{173\cdot 6 \times 230 \times 175}$

to find r.

Here $\dfrac{K - K'}{K} = \dfrac{112}{175F}$.

$$\begin{array}{rl}
\log 173\cdot 6 = & 2\cdot 2395 \\
\log 230 = & 2\cdot 3617 \\
\operatorname{colog} 295 = & \bar{3}\cdot 5302 \\
\hline
\log F = & \underline{2\cdot 1314} \\
\operatorname{colog} F = & \bar{3}\cdot 8686 \\
\operatorname{colog} 175 = & \bar{3}\cdot 7570 \\
\log 112 = & 2\cdot 0492 \\
\hline
 & \bar{3}\cdot 6748
\end{array}$$

$$\frac{K - K'}{K} = \cdot 00473$$

$$\frac{K'}{K} = \cdot 9953.$$

$$\begin{array}{rl}
\log \dfrac{K'}{K} = & \bar{1}\cdot 9979 \\
\log \dfrac{K}{K'} = & \cdot 0021 \\
\hline
\operatorname{colog}\left(\log \dfrac{K}{K'}\right) = & 2\cdot 6778 \\
\operatorname{colog} F = & \bar{3}\cdot 8686 \\
\log 90 = & 1\cdot 9542 \\
\log \cdot 4343 = & \bar{1}\cdot 6378 \\
\hline
\log r = & 2\cdot 1384 \\
r = & \underline{137\cdot 5}
\end{array}$$

(v) The equation—

$$P = a + b\,D^m$$

having been assumed as connecting the pressure per square foot of platen, and D in lbs. per c. ft. the density of the material compressed in a press, a, b and m being constants as far as an approximate solution is concerned, required to

find their values, it being given that the values 70, 50, 25 of D correspond to the values 190, 50, 6·5 of P.

Industries, vol. vi., p. 385.

We have $190 = a + b\ 70^m$ (1)

$$50 = a + b\ 50^m \quad (2)$$

$$6{\cdot}5 = a + b\ 25^m \quad (3)$$

For a first approximation assume that in equations (1) and (2) a is small compared with the numbers on the left hand. This gives—

$$190 = b\ 70^m$$

$$50 = b\ 50^m$$

$$\tfrac{19}{5} = (\tfrac{7}{5})^m$$

$$m\ (\log 7 - \log 5) = \log 19 - \log 5$$

$$\begin{array}{lcl} {\cdot}8451 & & 1{\cdot}2788 \\ \underline{{\cdot}6990} & & \underline{{\cdot}6990} \\ {\cdot}1461\ m & = & {\cdot}5798 \end{array}$$

$$m = 4.$$

$$\log b = \log 190 - 4 \log 70.$$

$$\begin{array}{rl} & 2{\cdot}2788 \\ & \underline{1{\cdot}8451}\ (\times\ 4) \\ \log b = & \bar{6}{\cdot}8984 \\ \log 25 = & \underline{1{\cdot}3979}\ (\times\ 4) \\ 3{\cdot}1 & {\cdot}4900 \end{array}$$

$$a = 3{\cdot}4$$

For a second approximation substitute this value of a.

$$186{\cdot}6 = b\ (70)^m$$

$$46{\cdot}6 = b\ (50)^m$$

$$m\ (\log 7 - \log 5) = \log 186{\cdot}6 - \log 46{\cdot}6$$

$$\begin{array}{rl} & 2{\cdot}2709 \\ & \underline{1{\cdot}6684} \\ {\cdot}1461m = & {\cdot}6025 \end{array}$$

$$m = 4{\cdot}12$$

$$\begin{array}{r} 1461\ )\ 6025 \\ 181 \\ 35 \end{array}$$

$$\log b = \log 186{\cdot}6 - 4{\cdot}12 \log 70$$

$$\begin{array}{r}
1{\cdot}8451 \\
\underline{4{\cdot}12} \\
7{\cdot}3804 \\
{\cdot}1845 \\
\underline{{\cdot}0369} \\
7{\cdot}6018 \\
\underline{2{\cdot}2709}
\end{array}$$

$$\log b = \bar{6}{\cdot}6691$$

$$a = 6{\cdot}5 - b\,(25)^{4{\cdot}12}$$

$$\begin{array}{r}
1{\cdot}3979 \\
\underline{4{\cdot}12} \\
5{\cdot}5916 \\
{\cdot}1398 \\
\underline{{\cdot}0279} \\
5{\cdot}7593 \\
\underline{\bar{6}{\cdot}6691} \\
{\cdot}4284
\end{array}$$

$$\begin{array}{r}
2{\cdot}68 \\
\underline{6{\cdot}5}
\end{array}$$

$$a = 3{\cdot}82$$

ANSWERS.

1. 47326406.
2. 447172204.
3. 628208761.
4. (i) 17923602; (ii) 9206740867.
5. 818442.
6. £333002 17s. 5d.
7. 17861642575.
8. 166859352.
9. 166745684.
10. 288000158.
11. 621605194.
12. 1274866327776.
13. 964832407928.
14. 2218758178733.
15. 18204196512535—13745739598098.
16. 70251807402.
17. 121932631112635269.
18. 23884044718.
19. 785770847493237225.
20. 4012099469.
21. 579.
22. 7·312.
23. (i) 147; (ii) 9·09375.
24. (i) 87; (ii) 481; (iii) 4371.
25. (i) 8·25; (ii) 5·125; (iii) 3·754.
26. 397; 4768; 19075.
27. ·102452; ·1035497; ·10373265.
28. (i) 1119815176229434489; (ii) 99511134896917780; (iii) 80033949888975; (iv) 7457780266084816245.
29. (i) 694088065824327; (ii) 674821925209773; (iii) 654578637545784.
30. (i) 656565818662610; (ii) 694178189398890.
31. 188·372899.
32. 983596580460.
33. 1643·02591971.

34. (i) 218423026350; (ii) 2730287829375.

35. (i) ·841575; (ii) 4·207875.

36. (i) 1851851835; (ii) 3086419725.

37. (i) 7898529563; (ii) 79783137; (iii) 79863.

38. (i) 4432185825; (ii) 295479055.

39. (i) 579; (ii) $\frac{10816}{78469}$.

40. (i) 700569; (ii) 837.

41. (i) 87·938; (ii) 15972·6.

42. ·1058.

43. ·027.

44. (i) 845763; (ii) 73255524 . . . 693; (iii) 845763.

45. (i) 58372617; (ii) 305313700; (iii) 5837·2617.

46. (i) $\frac{7}{9}$; (ii) $\frac{45}{73}$; (iii) $\frac{2}{3}$; (iv) $\frac{31560}{42441}$; (v) $\frac{207}{464}$.

47. (i) $2^7, 3^2, 5, 7$; (ii) $2^6, 3^4, 7, 11$; (iii) $2^8, 3^5, 7, 11$; (iv) $2^{18}, 3^{10}, 11^2, 13, 17, 19, 23$.

48. (i) $\frac{403}{571}$; (ii) $\frac{5}{13}$.

49. 3·1416.

50. (i) 33467875; (ii) ·00013602875.

51. $x^6 - 10x^5 + 25x^4$; $x^6 - 10x^5 + 33x^4 - 40x^3 + 16x^2$; $x^6 - 10x^5 + 33x^4 - 46x^3 + 46x^2 - 24x + 9$.

52. $x^6 + 6x^5 + 17x^4 + 34x^3 + 46x^2 + 40x + 25$.

53. (i) 7021266849; (ii) 7019423524; (iii) 7189683264.

54. 973182·25.

55. 975223·401156; 975215·5009.

56. 325476.

57. 323765.

58. 648099.

59. 646392.

60. 96974614.

61. 1014310.

62. 14794.

63. 161908617.

64. 27723103.

65. 5385071·0625.

66. 93557·652447.

67. 6705275·08995727.

68. $x^9 - 15x^8 + 75x^7 - 125x^6$; $x^9 - 15x^8 + 87x^7 - 245x^6 + 348x^5 - 240x^4 + 64x^3$; $x^9 - 15x^8 + 87x^7 - 254x^6 + 438x^5 - 537x^4 + 451x^3 - 279x^2 + 108x - 27$.

69. $x^9 + 9x^8 + 39x^7 + 114x^6 + 246x^5 + 399x^4 + 499x^3 + 465x^2 + 300x + 125$.

71. 588333013078257.

72. 588101341687768; 609627623321088.

73. 960044289·625.
74. (i) 963054563·603777 ; (ii) 963066266·237189304.
75. 185195848.
76. 183574758.
77. 575119943.
78. (i) 356·016472217 ; (ii) 126·247641263.
79. 57·049719232.
80. (i) 381268·676 ; (ii) 589·683026 ; (iii) 10·209013526.
81. 532·543309041755.
82. (i) 48 ; (ii) 1534 ; (iii) 3265 ; (iv) 4778.
83. (i) 253 ; (ii) 2345 ; (iii) 2461 ; (iv) 3572 ; (v) 23 ; (vi) 21·75.
84. (i) 123 ; (ii) 23·4 ; (iii) 3·45 ; (iv) ·456 ; (v) ·0567.
85. ·0000008 ; ·0000009.
86. (i) ·000000085 ; (ii) ·00000000012.
87. (i) ·98968853 ; (ii) ·00000219 ; (iii) ·00000222.
88. (i) ·000014 ; (ii) ·0017.
89. (*a*) (i) 9·0100 ; (ii) 9·8696 ; (iii) 13·292 ; (iv) 12·775 ; (v) 7·4162.
(*b*) (i) 705518·8 ; (ii) 705518·7 ; (iii) 705518·7.
92. (i) 9·6139 ; (ii) 94·8853 ; (iii) ·0095 ; (iv) 2·5000 ; (v) 17·101.
93. (i) 559 ; (ii) 594·35 ; (iii) 607·5 ; (iv) 70·3715 ; (v) 26693·4.
94. 12·92995.
95. (i) 42·41326694 ; (ii) 3·560753934 ; (iii) 1421836·886 ;
(iv) 39422000000000000 ; (v) 188·7115.
96. (i) 2·3981 ; (ii) ·81650 ; (iii) 583·77 ; (iv) 14·142 ; (v) 3·8540.
97. (i) 2·2361 ; (ii) 9·500 ; (iii) 1·0417 ; (iv) ·1117 ; (v) 291·7190.
98. 70·711.
99. 304·7.
100. ·0887.
101. ·0103093.
102. ·00100300902708124.
103. ·0001000200040008001600320064.
104. Quotients (i) 1, 1, 1768 ; (ii) 4, 3, 1, 4, 1, 2, 1, 11, 2, 6 ;
(iii) 5, 1, 1, 3, 5, 3, 1, 1, 3, 249, 4 ;
(iv) 1, 32, 1, 1, 2, 2, 3, 1, 8, 3, 1, 3, 27, 1, 3, 11.
105. $\frac{4}{1}, \frac{13}{3}, \frac{17}{4}, \frac{81}{19}, \frac{98}{23}, \frac{277}{65}$; $\frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}, \frac{657}{118}$; $\frac{1}{1}, \frac{33}{32}, \frac{34}{33}, \frac{67}{65}, \frac{168}{163}, \frac{403}{391}$.
106. 7021.
107. 7021000000.
108. 9147.
109. (i) 914770000000 ; (ii) 7019·4 ; (iii) 718950.
110. 1606.
111. (i) 18 ; (ii) 29.
112. 95.

N

113. 50·05.
114. 2847200.
115. (i) 5585071; (ii) 93558; (iii) 6705175.
116. 9629.
117. 960000.
118. (i) 963·0546; (ii) 963·0663.
119. (i) 223·1; (ii) 208·9.
120. 359·7.
121. (i) 290·7; (ii) 290·6539.
122. (i) 1·41421356237; (ii) 1·73205080757; (iii) 5·29150262213; (iv) 5·56776436283.
123. (i) ·4636477; (ii) ·3217505.
124. (i) 1·1925824; (ii) 3·8; (iii) 30·29; (iv) 3·429.
125. (i) 2·410; (ii) 2·41014; (iii) 2·410142.
126. (i) 2·080; (ii) 2·08008; (iii) 2·08008383.
127. (i) 8·86; (ii) 8·86437; (iii) 8·86437165; (iv) 8·86437165393.
128. 4·10283.
129. (i) 1·0280; (ii) 5·879; (iii) 3·3622; (iv) 4·302; (v) 5·1349.
130. 8·124.
131. 1·098612.
132. $\bar{1}$·2394997; $\bar{4}$·2394997; 7·2394997.
133. $\bar{4}$·4789994; $\overline{14}$·1974985; $\bar{1}$·4131666; $\bar{1}$·2478999.
134. 3·3010300; $\bar{3}$·3010300; $\bar{1}$·3010300; 3·0103; 4·4771213; $\bar{2}$·4771213; 47·71213.
135. (i) 16; (ii) 48.
136. (i) 2·3530888; (ii) $\bar{3}$·3536085.
137. (i) 2253·1; (ii) $\bar{3}$·3536085.
138. (i) 1·3534912; (ii) $\bar{4}$·3538652.
139. (i) ·00000225836; (ii) 2254156.
140. (i) 5·099457; (ii) $\bar{1}$·097840.
141. (i) ·000125304; (ii) 1259·68.
142. (i) 1·56993; (ii) 4·56825; (iii) $\bar{3}$·56944.

143.

(1)	1·2
(2)	2·4
(3)	3·6
(4)	4·8
(5)	6·0
(6)	7·2
(7)	8·4
(8)	9·6
(9)	10·8.

144. (i) ·7435; (ii) 2·7451; (iii) $\bar{3}$·7466; (iv) 2·7409; (v) $\bar{2}$·7462.
145. (i) ·02096; (ii) 208; (iii) 002089; (iv) ·0002101.
146. 62·112.
147. 86·859.
148. ·000000000479715.
149. 1·244732.
150. (i) $\bar{4}$·6461414; (ii) $\bar{1}$·6470076; (iii) 3·6477017.
151. (i) $\bar{6}$·43133; (ii) 4·42992; (iii) $\bar{8}$·2541; (iv) 1·2581.
152. (i) ·031417; (ii) ·99649; (iii) ·00550662; (iv) ·960404.
153. (i) ·421501; (ii) ·0011649.

154. (i) 6·58684; (ii) 5·41769; (iii) 3·67376; (iv) 2·55578.
155. (i) 22·545; (ii) 1·8508; (iii) 1·47515.
156. (i) $\bar{1}$·6866179; (ii) $\bar{1}$·9414978; (iii) $\bar{1}$·7453592; (iv) ·0586580; (v) ·3142199.
157. (i) 29° 3′ 26″·4; (ii) 29° 3′ 1″·1; (iii) 29° 1′ 31″.
158. (i) $\bar{1}$·789269; (ii) $\bar{1}$·892745; (iii) $\bar{1}$·789375; (iv) ·107372.
159. (i) 37° 59′ 7″; (ii) 37° 59′ 18″; (iii) 59° 0′ 5″; (iv) 52° 0′ 27″.
164. (i) 318·47; (ii) 209·61; (iii) 358·53; (iv) 441·93; (v) 186·19.
165. 682·955.
166. (i) 1000·509; (ii) 303·128; (iii) 183·892; (iv) 495·470.
167. (i) 584·062; (ii) 7·44665; (iii) 38805.
168. (i) a = 149·47, b = 418·51; (ii) b = 11·241, c = 9·6697; (iii) a = 18448, c = 15965; (iv) b = 7000, c = 8999.
169. (i) 60° 13′ 48″ or 119° 46′ 12″; (ii) 45° 8′ 38″ or 134° 51′ 22″; (iii) 20° 10′ 43″ or 159° 49′ 17″; (iv) 36° 0′ 29″ or 143° 59′ 31″.
170. (i) (α) B = 64° 16′ 11″; c = 843·317.
(β) B = 115° 43′ 49″; c = 510·00.
(ii) (α) C = 21° 26′ 14″; a = 96·4775.
(β) C = 158° 33′ 46″; a = 3·95426.
(iii) (α) C = 50° 1′ 40″; b = 10·2535.
(β) C = 129° 58′ 20″; b = 3·32602.
171. (i) A = 70° 31′ 33″·5; B = 55° 24′ 7″·5.
(ii) B = 91° 39′ 13″; C = 68° 20′ 47″.
(iii) C = 87° 35′ 36″; A = 44° 22′ 52″.
(iv) B = 160° 17′ 36″·6; C = 9° 42′ 23″·4.
172. a = 3650·3.
173. (i) c = 3224·3; (ii) a = 14·2312; (iii) b = 3·4844.
174. (i) B = 106° 7′ 28″; A = 39° 44′ 32″.
(ii) B = 40° 22′ 37″·5; C = 77° 28′ 7″·5.
(iii) A = 97° 33′ 59″; B = 41° 6′ 34″.
(iv) A = 111° 38′ 14″; C = 15° 7′ 34″.
175. (i) A = 23° 42′ 19″; B = 27° 55′ 0″; C = 128° 22′ 41″.
(ii.) A = 41° 24′ 34″·5; B = 55° 46′ 16″·5; C = 82° 49′ 9″.
(iii) A = 49° 25′ 52″; B = 83° 22′ 30″; C = 47° 11′ 38″.
(iv) A = 21° 47′ 12″; B = 38° 12′ 48″; C = 60°.
181. A = 114° 54′ 29″·1; B = 11° 12′ 39″·6; C = 53° 52′ 51″·3.
182. ·92584 – 37·468.
183. 3524·21; – 16·21.
184. – 15·14843; – 156·78157.
185. 9·87317; 493·3868.
186. (i) 4·4058; ·3351; – 4·710; (ii) – 3·97196; (iii) 1·3569; 1·6920; – 3·0489; (iv) 5·3569; 5·6920; ·9511.
187. – 1·73205; – 2·64575; 4·3778.

188. $-2{\cdot}09455$.
189. $2{\cdot}426$.
190. $7{\cdot}6688$.
191. $x = 9{\cdot}561$.
192. $3{\cdot}83266$.
193. $\theta = 77^\circ\ 51'\ 51''{\cdot}3$ or $21^\circ\ 36'\ 4''{\cdot}1$.
194. $x = 12{\cdot}5491$ or $1{\cdot}61143$.
195. $x = 0111$.
196. $x = 4{\cdot}22685$.
197. $u = {\cdot}00022289$.
198. $109^\circ\ 6'\ 23''{\cdot}8$.
199. $57^\circ\ 34'\ 42''$; $82^\circ\ 25'\ 18''$.
200. (i) $x = 1{\cdot}0557$. (ii) Putting $x^2 = 1$, we get $x = 1{\cdot}05$; putting $x^2 = (1{\cdot}05)^2$, we get $x = 1{\cdot}055$; putting $x^2 = (1{\cdot}055)^2$, we get $x = 1{\cdot}0556$; putting $x^2 = (1{\cdot}0556)^2$, we get $x = 1{\cdot}0557$.

TABLE I.

Factors for effecting various Reductions with their Logarithms.*

		Factors.	Logs.
1.		$\pi = 3·1415927$	·497150
	(i) $3 + \frac{1}{7} - \frac{1}{800}$	$= 3·1416\frac{1}{14}$	
	(ii) $3 + \frac{1}{8} + \frac{1}{60}$	$= 3·1416\frac{2}{3}$	
	(iii) $(3 + \frac{1}{7})(1 - ·0004)$	$= 3·1416$	
	(iv) $3(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= 3·1416$	
	(v) $3 + \frac{1}{7} - \frac{1}{800} - \frac{1}{70000}$	$= 3·141592\frac{4}{7}$	
2.		$2\pi = 6·2831853$	·798180
	(i) $6 + \frac{1}{4} + \frac{1}{30}$	$= 6·2833\frac{1}{3}$	
	(ii) $6(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= 6·2832$	
3.		$4\pi = 12·566371$	1·099210
	(i) $12 + \frac{1}{2} + \frac{1}{20} + \frac{1}{60}$	$= 12·5666\frac{2}{3}$	
	(ii) $12(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= 12·5664$	
4.		$\frac{1}{3}\pi = 1·0471976$	·020029
	(i) $(1 + \frac{1}{20})(1 - \frac{1}{400})$	$= 1·0473\frac{3}{8}$	
	(ii) $(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= 1·0472$	
		$\frac{\pi}{4} = ·785398$	$\bar{1}·895089$
	$·77(1·02) = ·77(1 + \frac{1}{50})$	$= ·7854$	
6.		$\frac{1}{6}\pi = ·5235988$	$\bar{1}·718999$
	(i) $(1 + \frac{1}{20})(\frac{1}{2} - \frac{1}{800})$	$= ·5236\frac{9}{10}$	
	(ii) $(1 + \frac{1}{20})(\frac{1}{2} - \frac{1}{800} - \frac{1}{12000})$	$= ·5236$	
7.		$\frac{4}{3}\pi = 4·1887902$	·622089
	(i) $(1 + \frac{1}{20})(4 - \frac{1}{100})$	$= 4·1895$	
	(ii) $4(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= 4·1888$	
8.		$\frac{1}{\pi} = ·31830989$	$\bar{1}·502850$
	(i) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200})$	$= ·3183\frac{1}{3}$	
	(ii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200} + \frac{1}{50000})$	$= ·31831\frac{1}{3}$	
	(iii) $\frac{1}{3} - (\frac{1}{100} + \frac{1}{200} + \frac{1}{40000})$	$= ·318308\frac{1}{3}$	

*For the application of the methods of "Practice" for effecting reductions and multiplication, see pp. 24, 80, 81.

9.		$\pi^2 = 9{\cdot}86960440$	$\cdot994300$
	$10(1 - \cdot01304)$	$= 9{\cdot}8696$	
10.		$\frac{1}{\pi^2} = \cdot1013212$	$\bar{1}{\cdot}005700$
	$\frac{1}{10}(1 + \cdot01321)$	$= \cdot101321$	
11.		$\sqrt{\pi} = 1{\cdot}77245385$	$\cdot248575$
	(i) $2(1 - \frac{1}{9})$	$= 1{\cdot}77\frac{7}{9}$	
	(ii) $2(1 - \frac{1}{9} - \frac{1}{360})$	$= 1{\cdot}772\frac{2}{9}$	
12.		$\sqrt[3]{\frac{\pi}{6}} = \cdot80599598$	$\bar{1}{\cdot}906333$
	$1 - (\frac{1}{6} + \frac{1}{36})$	$= \cdot805\frac{5}{9}$	
13.		$\sqrt{2} = 1{\cdot}41421356$	$\cdot1505150$
	$1{\cdot}4(1 + \frac{1}{100} + \frac{1}{10000} + \frac{1}{20000})$	$= 1{\cdot}41421$	
14.		$\sqrt{3} = 1{\cdot}73205081$	$\cdot238561$
	$1 + \frac{1}{2} + \frac{1}{4}$	$= 1{\cdot}75$	
	$(1 + \frac{1}{2} + \frac{1}{4})(1 - \frac{1}{100})$	$= 1{\cdot}7325$	
15.		$\sqrt[3]{2} = 1{\cdot}259921$	$\cdot100343$
	$1 + \frac{1}{4} + \frac{1}{100}$	$= 1{\cdot}26$	
16.		$\sqrt[3]{3} = 1{\cdot}442250$	$\cdot159040$
	$1{\cdot}4 \times 1{\cdot}03$	$= 1{\cdot}442$	
17.	Degrees to radians	$\cdot01745329$	$\bar{2}{\cdot}241877$
	$\frac{1}{60}(1 + \frac{1}{20})(1 - \frac{1}{400} - \frac{1}{6000})$	$= \cdot01745333$	
18.	Radians to degrees	$57{\cdot}29578$	$1{\cdot}758123$
	$60(1 - \frac{1}{20} + \frac{1}{200})$	$= 57{\cdot}3$	
19.	Minutes to radians	$\cdot00029088$	$\bar{4}{\cdot}463717$
	$\cdot0003(1 - \cdot03)$	$= \cdot000291$	
20.	Seconds to radians	$\cdot000004848$	$\bar{5}{\cdot}685566$
21.	Nap. to Common Logs	$\mu = \cdot43429448$	$\bar{1}{\cdot}637784$
	$\cdot43(1 + \frac{1}{100})$	$\cdot4343$	
	$\cdot43(1 + \frac{1}{100})(1 - \frac{1}{80000})$	$= \cdot43429457$	
24.	Com. to Nap Logs	$\frac{1}{\mu} = 2{\cdot}302585$	$\cdot362216$
	$2{\cdot}3(1 + \frac{1}{900})$	$= 2{\cdot}3025\frac{5}{9}$	
25.	Metres to yds.	$1{\cdot}093623$	$\cdot038868$
	$1 + \frac{1}{12} + \frac{1}{96}$	$= 1{\cdot}09375$	
26.	Yds. to metres	$\cdot9143918$	$\bar{1}{\cdot}961132$
	$1 - \frac{1}{10} + \frac{1}{70}$	$= \cdot914\frac{2}{7}$	
27.	Metres to ft.	$= 3{\cdot}280869$	$\cdot515989$
	$3 + \frac{1}{4} + \frac{1}{32}$	$= 3{\cdot}28125$	
28.	Ft. to metres	$\cdot304797$	$\bar{1}{\cdot}484011$
	$\cdot3(1 + \frac{1}{60})$	$= \cdot305$	